We want to compute $\langle 0 | T\, \phi(x)\, \phi(y) | 0 \rangle$

(1.) $x^0 > y^0$:

$$\langle 0 | \phi(x)\, \phi(y) | 0 \rangle = \langle 0 | \phi(x)\, \mathbb{1}\, \phi(y) | 0 \rangle$$

with $\mathbb{1} = |0\rangle\langle 0| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} |\lambda_p\rangle\langle\lambda_p|$

$E_p(\lambda) = \sqrt{|\vec{p}|^2 + m_\lambda^2}$

$$= \underbrace{\langle 0 | \phi(x) | 0 \rangle}_{=0} \langle 0 | \phi(y) | 0 \rangle + \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} \langle 0 | \phi(x) | \lambda_p \rangle \langle \lambda_p | \phi(y) | 0 \rangle$$

$$\langle 0 | \phi(x) | \lambda_p \rangle = \langle 0 | \underbrace{e^{i\hat{P}x}\, \phi(0)\, e^{-i\hat{P}x}}_{=\phi(x) \text{ see } (2.49)\ [PS]} | \lambda_p \rangle$$

$$\langle 0 | \hat{P}_\mu = (\hat{P}_\mu^\dagger |0\rangle)^\dagger = (\hat{P}_\mu |0\rangle)^\dagger = 0^\dagger = 0$$

$\hookrightarrow \langle 0 | \hat{P} = 0$ and $\langle 0 | e^{i\hat{P}x} = \langle 0 |$

$\hat{P}_\mu |\lambda_p\rangle = P_\mu |\lambda_p\rangle$, therefore

$e^{-i\hat{P}x} |\lambda_p\rangle = e^{-ipx} |\lambda_p\rangle \Big|_{p^0 = E_p}$ because $\hat{P}_0 |\lambda_p\rangle = E_p$

Initial assumption Lorentz invariance

$\phi(0) = U_p\, \phi(0)\, U_p^{-1}$ and

$\langle 0 | U_p = \langle 0 |$ or $(\langle 0 | U_p)^\dagger = 0 = \underbrace{(U_p)^\dagger}_{U_p^{-1} \text{ (unitary)} = U_{-p}} |0\rangle = 0$

$$= \langle 0 | \phi(0) | \lambda_p \rangle e^{-ipx} \Big|_{p^0 = E_p}$$

$$= \langle 0 | U_p\, \phi(0)\, U_p^{-1} | \lambda_p \rangle e^{-ipx} \Big|_{p^0 = E_p}$$

but $U_{p'} |\lambda_p\rangle = |\lambda_{p+p'}\rangle$

$$= \langle 0| \phi(0) |\lambda_0\rangle e^{-ipx} \Big|_{p^0 = E_p}$$

$$\langle 0| \phi(x)\, \phi(y) |0\rangle = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} \cdot |\langle 0| \phi(0) |\lambda_0\rangle|^2 \, e^{-ip(x-y)} \Big|_{p^0 = E_p}$$

$$= \sum_\lambda |\langle 0| \phi(0) |\lambda_0\rangle|^2 \, D(x-y)$$ see (2.50) [PS]

You can now follow [PS] to see how to get $D_F(x-y)$ in (2.59) of [PS] after taking into account time ordering.