Revision notes on Classical Field Theory

January 2024

The present constitutes a short summary of what was discussed in the class during the final tutorial of the Classical Field Theory course.

The Lorentz group in 2+1 dimensions

The main subject of discussion was the Lorentz group in $d = 3$ dimensions. Just as in the 2d case, we are searching for transformation matrices that satisfy the equations

$$\Lambda^T \cdot \eta \cdot \Lambda = \eta, \tag{1}$$

$$\det \Lambda = \pm 1, \tag{2}$$

where

$$\eta_{\mu\nu} = diag(1, -1, -1).$$

We can do so by brute force; that is plugging the most general form a 3×3 matrix may assume:

$$\Lambda = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix},$$

in (1) and solve the system of equations to determine the parameters. Doing so, the resulting system reads

$$\begin{cases} a^2 - d^2 - g^2 = 1 \\ b^2 - e^2 - h^2 = -1 \\ c^2 - f^2 - k^2 = -1 \\ ab - de - gh = 0 \\ ac - df - gk = 0 \\ bc - ef - hk = 0 \\ aek + bfg + cdh - ceg - bdk - afh = \pm 1 \end{cases} \tag{3}$$

This system is of course very difficult to solve. Therefore, we will do a small trick by not actually solving it.

First of all, we have to think what information does the 3d Lorentz group provide us with. In general, the Lorentz group gives rise to boosts and rotations and in $d = 3 = 2 + 1$ particularly we have time and 2 spatial dimensions, namely x and y; therefore we have <u>2 boosts and 1 rotation.</u>

Secondly, the symmetry group of the 3d Lorentz group is $SO(2, 1)$ whose generators are

$$T_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

This means that each of the T_i's gives rise to either a boost or a rotation; problem is to understand which is which. The relative minus sign however in T_3 gives us a hint that it corresponds to a rotation and we can make the following identification:

- T_1 is the generator of boosts in $x-$axis, which are parametrized as
$$\Lambda^{(tx)} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 \\ -\sinh\phi & \cosh\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}; \tag{4}$$
- T_2 is the generator of boosts in $y-$axis which, are parametrized as
$$\Lambda^{(ty)} = \begin{pmatrix} \cosh\phi & 0 & -\sinh\phi \\ 0 & 1 & 0 \\ -\sinh\phi & 0 & \cosh\phi \end{pmatrix}; \tag{5}$$
- T_3 is the generator of rotations in the xy plane, which are parametrized as
$$\Lambda^{(xy)} = \begin{pmatrix} 1 & 0 & 0 \\ \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \end{pmatrix}, \tag{6}$$

  where we have used
$$T_i = \left.\frac{\partial \Lambda_i}{\partial \phi}\right|_{\phi=0}$$
  to carry out this computation. Notice also that we parametrize boosts using hyperbolic trigonometric functions and rotations using regular trigonometric functions. Interested readers are invited to check for themselves that these three matrices satisfy the system of (3)

However, one can easily notice that (4) - (6) have ${\Lambda_0}^0 > 0$ and $\det\Lambda = 1$, which means that they all correspond to proper-orthochronous transformation. So, how can one obtain improper and non-orthochronous transformations? In principle, one has to split each boost and each rotation into these components while always satisfying (3). Hence, the respective components of connectivity are the following:

i) *For $x-$boost:*

$\Lambda^{(tx)\uparrow}_{+} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 \\ -\sinh\phi & \cosh\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\Lambda^{(tx)\downarrow}_{+} = \begin{pmatrix} -\cosh\phi & \sinh\phi & 0 \\ \sinh\phi & \cosh\phi & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$\Lambda^{(tx)\uparrow}_{-} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 \\ -\sinh\phi & -\cosh\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\Lambda^{(tx)\downarrow}_{-} = \begin{pmatrix} -\cosh\phi & \sinh\phi & 0 \\ \sinh\phi & \cosh\phi & 0 \\ 0 & 0 & -1 \end{pmatrix}$

ii) *For $y-$boost:*

$\Lambda^{(ty)\uparrow}_{+} = \begin{pmatrix} \cosh\phi & 0 & -\sinh\phi \\ 0 & 1 & 0 \\ -\sinh\phi & 0 & \cosh\phi \end{pmatrix}$	$\Lambda^{(ty)\downarrow}_{+} = \begin{pmatrix} -\cosh\phi & 0 & \sinh\phi \\ 0 & -1 & 0 \\ -\sinh\phi & 0 & \cosh\phi \end{pmatrix}$
$\Lambda^{(ty)\uparrow}_{-} = \begin{pmatrix} \cosh\phi & 0 & \sinh\phi \\ 0 & 1 & 0 \\ -\sinh\phi & 0 & -\cosh\phi \end{pmatrix}$	$\Lambda^{(ty)\downarrow}_{-} = \begin{pmatrix} -\cosh\phi & 0 & \sinh\phi \\ 0 & -1 & 0 \\ \sinh\phi & 0 & -\cosh\phi \end{pmatrix}$

iii) *For rotation:*

$\Lambda^{(xy)\uparrow}_{+} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$	$\Lambda^{(xy)\downarrow}_{+} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & \sin\phi & -\cos\phi \end{pmatrix}$
$\Lambda^{(xy)\uparrow}_{-} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & -\sin\phi & -\cos\phi \end{pmatrix}$	$\Lambda^{(xy)\downarrow}_{-} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$

In the depiction above. upper rows correspond to *proper* transformations ($det\Lambda = 1$; preserve orientation) while lower rows to *improper* ones ($det\Lambda = -1$; don't preserve orientation) and left columns correspond to *orthochronous* transformations (${\Lambda_0}^0 > 0$; preserve causality) while right columns to *non-orthochronous* ones (${\Lambda_0}^0 < 0$; don't preserve causality). Having these matrices, we can do any kind of calculation we want.

One last thing left to do, is the geometrical interpretation of vectors of the same length. We know already that a Lorentz transformation leaves the length of a vector invariant:

$$v'^{\mu}v'_{\mu} = \Lambda^{\mu}{}_{\rho}v^{\rho}\Lambda_{\mu}{}^{\sigma}v_{\sigma} = \delta^{\sigma}_{\rho}v^{\rho}v_{\sigma} = v^{\rho}v_{\rho}.$$

This means that the square length of a vector remains constantly the same and for a $3-$component vector we can write

$$v^{\mu}v_{\mu} = v^2 = v_0^2 - v_1^2 - v_2^2 = l. \tag{7}$$

What remains now is to classify between the different values l may assume. We deduce that

- *For* $l > 0$, we get a hyperboloid of 2 sheets (this is where timelike vectors reside);
- *For* $l < 0$, we get a hyperboloid of 1 sheet (this is where spacelike vectors reside);
- *For* $l = 0$, we get a cone (this is where lightlike vectors reside);

These 3 separate cases are apparent in Fig.1. Notice that causality is preserved in the timelike case (a connection between the past and future cone is forbidden). As a better comparison with the 2D case, the projection in the $v_0 - v_1$ plane is depicted in Fig.2a), along with the complete Minkowski space in Fig.2b). Notice, as a final remark, that spacelike connections lie outside the light-cone.

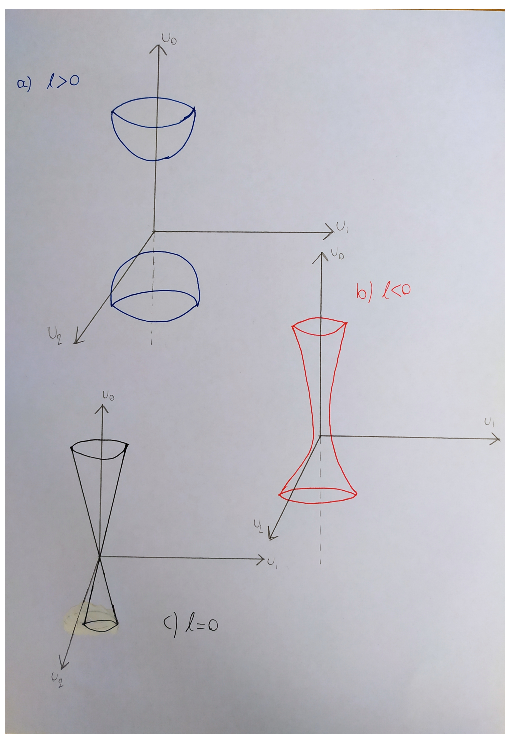

Figure 1: Geometrical interpretation of all possible sets of vectors in 3D Minskowski space.

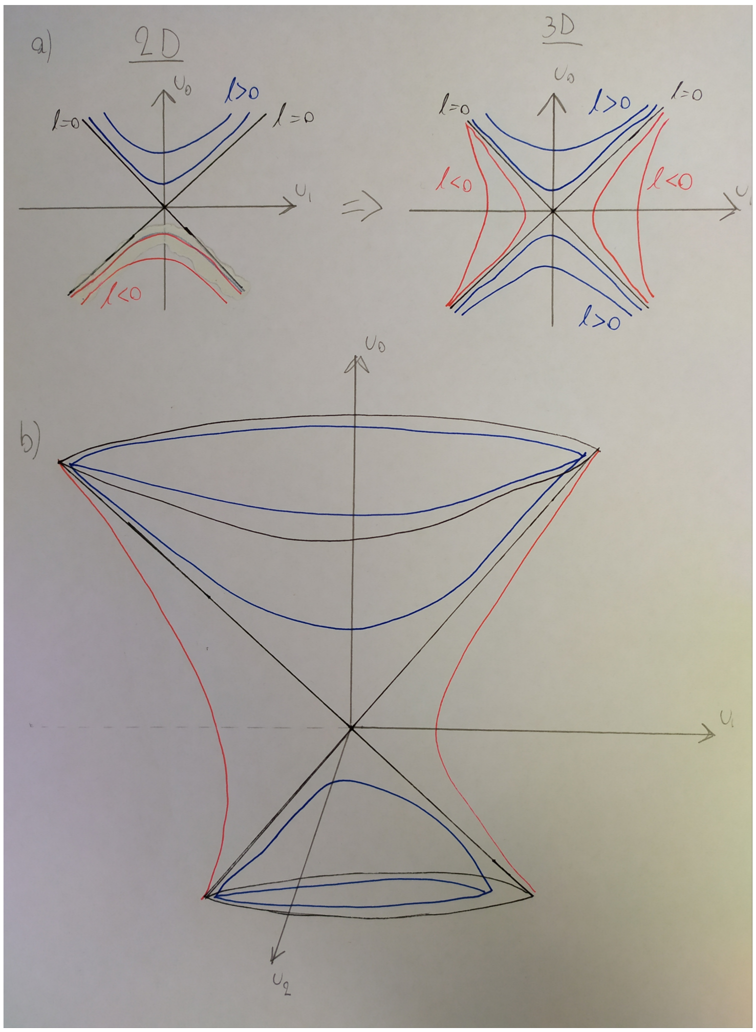

Figure 2: a) Comparison of $v_0 - v_1$ plane in 2D and 3D case b) Compact representation of 3D Minskowski space.