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Lie Algebras and Lie Groups – Practise Exam

16^{th} of January 2024

Please fill in:

Name: ____________________

Matriculation Number: ____________________

Number of Sheets: ____________________

Instructions – Please read carefully:

- Please write your full name and matriculation number on each sheet you hand in.
- Use a separate sheet of paper for each individual problem.
- You have 120 minutes to answer the questions.
- No resources are allowed.
- Use a blue or a black permanent pen.

Do not write below this line.

Comments:

Question:	1	2	3	4	Total
Points:	20	20	20	15	75
Score:					

1. General Properties of Lie Algebras and Lie Groups 20 points

(a) The structure coefficients of a Lie algebra g are defined by

$$[t_a, t_b] = f_{ab}{}^c t_c\,, \quad t_a, t_b \in g\,.$$

Derive their properties

i. (1 point) $f_{ab}{}^c = -f_{ba}{}^c$,

ii. (1 point) $f_{aa}{}^b = 0$, and

iii. (2 points) $\sum_c \left(f_{ab}{}^c f_{cd}{}^e + f_{da}{}^c f_{cb}{}^e + f_{bd}{}^c f_{ca}{}^e \right)$.

(b) To determine the dimension of $SO(N, \mathbb{R})$,

i. (1 point) Write down the defining properties of its group elements.

ii. (2 points) Use the exponential to obtain the properties of the corresponding generators.

iii. (1 point) Obtain the number of linearly independent generators.

(c) (1 point) Write down the definition of the Killing for $K(X, Y)$ for the Lie algebra generators $X, Y \in g$. And show that

i. (1 point) $K(X, Y) = K(Y, X)$ and

ii. (2 points) $K(X, [Y, Z]) = K([X, Y], Z)$.

(d) Consider the set of quaternions $G = \{\pm 1, \pm i, \pm j, \pm k\}$ and show that they

i. (4 points) form a finite group

ii. (1 point) that is non-commutative.

(e) (3 points) Show that a basis change of a real, N-dimensional vector space is captured by the Lie group $\mathrm{GL}(N, \mathbb{R})$.

2. Cartan-Weyl basis for B_n series 20 points

In the following we want to obtain an explicit matrix representation in the Cartan-Weyl basis for simple Lie algebras with B_n series Dynkin diagrams.

(a) (1 point) Draw the corresponding Dynkin diagram.

(b) (1 point) Which Lie algebra does it describe?

(c) (2 points) Read off the Cartan matrix A_{ij}.

(d) (4 points) Denote the Cartan generators as H_i, $i = 1, \ldots, n$, and the positive simple roots as E_i. Write the four relations that define the Lie algebra completely just in terms of A_{ij}.

(e) (2 points) For the next step, we need the $n \times n$-matrices $(\Sigma_{ab})_{cd} = \delta_{ac}\delta_{bd}$, calculate their commutators $[\Sigma_{ab}, \Sigma_{cd}]$.

(f) (2 points) Now combine them to the anti-symmetric matrices $A_{ab} = \Sigma_{ab} - \Sigma_{ba}$ and again compute their commutators.

(g) (2 points) Write down the Cartan generators in the fundamental representation in terms of A_{ab}'s.

(h) (4 points) Do the same for the positive simple roots.

(i) (2 points) How do you obtain the corresponding negative roots?

3. Irreducible representations

20 points

Here, we construct some irreducible representations of SO(5)

(a) (1 point) Draw the corresponding Dynkin diagram.

(b) (1 point) Read off the Cartan matrix A_{ij}.

(c) (1 point) From A_{ij} obtain all simple roots in the Dynkin basis.

(d) (3 points) Construct the weights arising form the highest weight $[1, 0]$.

(e) (1 point) What is the conjugate representation?

(f) (1 point) All these weights have multiplicity one. What is the dimension of this irrep?

(g) (5 points) Repeat (d)-(f) for $[0, 1]$.

(h) (3 points) What are the steps to compute the tensor product of these two irreps and decompose it into a sum of irreps?

(i) (4 points) Compute the multiplicity of the weight $[0, 1]$ in the weight system with the highest weight $[1, 1]$. *Hint: Use the Freundental reduction formula:*

$$\mathrm{mult}(\lambda) = \frac{2\sum_{\alpha>0}\sum_{m>0}(\lambda + m\alpha, \alpha)\mathrm{mult}_{\Lambda}(\lambda + m\alpha)}{(\Lambda + \rho, \Lambda + \rho) - (\lambda + \rho, \lambda + \rho)}$$

and $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$.

4. Quick questions 15 points

(a) (1 point) Draw the extended Dynkin diagram $\widehat{D}_n$.

(b) (1 point) Explain why we do not obtain a semisimple Lie algebra from it?

(c) (2 points) How do we obtain regular maximal subalgebras from it?

(d) (2 points) Draw the Young tableau of the fundamental and its conjugate irrep of SU(4).

(e) (2 points) Obtain their tensor product decomposition using these two Young tableaux.

(f) (1 point) What are the two major classes of symmetries in physics?

(g) (1 point) What is the gauge group of the standard model?

(h) (3 points) Which forces described by it constituents?

(i) (2 points) State GUT gauge groups which unify the standard model gauge groups?