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Selected Tools of Modern Theoretical Physics 2B – Practice Exam

16^{th} of June 2025

Please fill in:

Name: ______________________

Matriculation Number: ______________________

Number of Sheets: ______________________

Instructions – Please read carefully:

- Please write your full name and matriculation number on each sheet you hand in.
- Use a separate sheet of paper for each individual problem.
- You have 120 minutes to answer the questions.
- No resources are allowed.
- Use a blue or a black permanent pen.

Do not write below this line.

Comments:

Question:	1	2	3	4	Total
Points:	20	20	15	15	70
Score:					

1. General properties of Lie algebras and Lie groups 20 points

(a) The structure coefficients of a Lie algebra $\mathfrak{g}$ are defined by

$$[t_a, t_b] = f_{ab}{}^c t_c\,, \quad t_a, t_b \in \mathfrak{g}\,.$$

Derive their properties

i. (1 point) $f_{ab}{}^c = -f_{ba}{}^c\,,$

ii. (1 point) $f_{aa}{}^b = 0\,,$ and

iii. (2 points) $\sum_c \left(f_{ab}{}^c f_{cd}{}^e + f_{da}{}^c f_{cb}{}^e + f_{bd}{}^c f_{ca}{}^e\right) = 0\,.$

(b) To determine the dimension of $\mathrm{SO}(N, \mathbb{R})$,

i. (1 point) Write down the defining properties of its group elements.

ii. (2 points) Use the exponential map to obtain the properties of the corresponding generators of its Lie algebra.

iii. (1 point) Obtain the number of linearly independent generators.

(c) (1 point) Write down the definition of the Killing form $K(X, Y)$ for the Lie algebra generators $X, Y \in \mathfrak{g}$. And show that

i. (1 point) $K(X, Y) = K(Y, X)$ and

ii. (2 points) $K(X, [Y, Z]) = K([X, Y], Z)$.

(d) Consider the set of quaternions $G = \{\pm 1, \pm i, \pm j, \pm k\}$ and show that G

i. (4 points) forms a finite group under multiplication.

ii. (1 point) is non-commutative.

(e) (3 points) Show that a basis change of a real N-dimensional vector space is captured by the Lie group $\mathrm{GL}(N, \mathbb{R})$.

2. The Lie group SU(2) and its algebra 20 points

SU(2) is defined by 2×2, complex, unitary matrices with unit determinant.

(a) (2 points) Start with a general 2×2, complex matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

What are the three conditions for g to be a unitary matrix? Looking at the conditions you get, what is the real dimension of U(2)?

(b) (1 point) Now, additionally impose $\det(g) = 1$ to get SU(2). In this case, we can relabel the entries of g as $a = z_1$, $d = z_1^*$, $b = z_2$ and $c = -z_2^*$. Plug them into the conditions you obtained in the last task and verify that additionally

$$|z_1|^2 + |z_2|^2 = 1 \tag{1}$$

is needed.

(c) (1 point) Now use $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$, where all the x_i are real numbers. This allows you to show the equivalence of SU(2) with the unit three-sphere S^3. Explain how?

(d) (4 points) Next, we would like to compute the left-invariant Maurer-Cartan form. To this end, we first need a parameterization. Hopf coordinates,

$$z_1 = e^{i\xi_1} \sin\eta \qquad\qquad z_2 = e^{i\xi_2} \cos\eta$$

are a convenient choice. First show that they satisfy (1), and then compute $\omega = g^{-1}\mathrm{d}g$.

(e) (2 points) The unit element of SU(2) is just the unit matrix $g = \mathbf{1}$. Read off the corresponding values for the coordinates (ξ_1, ξ_2, η), and then compute the Maurer-Cartan form ω at this point. Obtain the generators for the Lie algebra $\mathfrak{su}(2)$ from this result. Compare them with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \text{and} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(f) (2 points) Verify that the Maurer-Cartan form satisfies the Maurer-Cartan equation, namely

$$\mathrm{d}\omega + \omega \wedge \omega = 0\,.$$

(g) (3 points) Define the adjoint action $\mathrm{Ad}_g(X) = gXg^{-1}$ for $g \in \mathrm{SU}(2), X \in \mathfrak{su}(2)$. Show that this map preserves the Killing form, and conclude that Ad_g defines an orthogonal transformation on $\mathfrak{su}(2)$.

(h) (3 points) Argue that this defines a group homomorphism $\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{SO}(3)$. What is its kernel?

(i) (2 points) From the above, explain why SU(2) is the double cover of SO(3), and what this means geometrically and/or algebraically (answer briefly but clearly).

3. Representation theory of $\mathfrak{su}(4)$ 15 points

Consider the Lie algebra $\mathfrak{su}(4)$ with the simple, positive, roots

$$\alpha_1 = \boxed{2 \;\; -1 \;\; 0}\,,$$
$$\alpha_2 = \boxed{-1 \;\; 2 \;\; -1}\,, \quad \text{and}$$
$$\alpha_3 = \boxed{0 \;\; -1 \;\; 2}\,.$$

(a) (1 point) What is the rank of this Lie algebra?

(b) (2 points) Take the Young tableau $\ydiagram{2,1}$ and determine its dimension. What specific irrep of $\mathfrak{su}(4)$ does it describe?

(c) (1 point) Explain how to read of highest weight $\Lambda = \boxed{210}$ from the Young tableau.

(d) (3 points) Construct the weight system by starting from this highest weight.

(e) (2 points) Explain how the root system and weight diagram of this representation are related.

(f) (1 point) What is the multiplicity of the weight $\boxed{0 \;\; 0 \;\; 0}$? *Hint: You do not need the Freudental reduction formula here. Rather remember the special role this weight plays.*

(g) (1 point) Find the dual representation.

(h) (3 points) Using the Killing form of $\mathfrak{su}(4)$ (we use the standard normalization where the roots have length 2), compute the Cartan matrix A_{ij}.

(i) (1 point) Another important representation is the fundamental representation $\square$. What is its dimensions expressed in terms of the rank?

4. Quick questions 15 points

(a) (1 point) What does the rank of a simple Lie algebra describe?

(b) (1 point) According to Levi decomposition, every finite-dimensional Lie algebra can be decomposed into two parts. What are they?

(c) (2 points) How are semisimple Lie algebras defined?

(d) (2 points) Draw the Young tableau of the fundamental and its conjugate irrep of SU(4).

(e) (2 points) Take the irreducible representation of SU(N) captured by the Young tableau $\ydiagram{2,1}$. What is its dimension?

(f) (1 point) What is the relevance of Lie groups in physics? Please be brief – you will get at most one point.

(g) (1 point) What is the difference between an Abelian and non-Abelian group?

(h) (3 points) What is a root in the context of a Lie algebra? Why are simple roots distinguished?

(i) (2 points) What is the adjoint representation of a Lie algebra? What is its dimension?