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Classical Field Theory – Practice Exam

4$^{\text{th}}$ of February 2026

Please fill in:

Name: ______________________________

Matriculation Number: ______________________________

Number of Sheets: ______________________________

Instructions – Please read carefully:

- Please write your full name and matriculation number on each sheet you hand in.
- Use a separate sheet of paper for each individual problem.
- You have 120 minutes to answer the questions.
- No resources are allowed.
- Use a blue or a black permanent pen.
- **You only have to solve 3 out of the 4 problems. Please indicate clearly the problems you selected. Only these will be graded.**

Do not write below this line.

Comments:

Question:	1	2	3	4	Total
Points:	15	15	15	15	60
Score:					

1. Scalar Electrodynamics 15 points

Here we consider a complex Klein-Gordon field whose global U(1) symmetry is gauged to obtain scalar electrodynamics, governed by the action

$$S = \int \mathrm{d}^4 x \left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{D_\mu \Phi} D^\mu \Phi - m^2 \overline{\Phi} \Phi \right) \,,$$

which is expressed in terms of the covariant derivative

$$D_\mu \Phi = (\partial_\mu + ieA_\mu)\Phi \,.$$

Moreover, there is the field strength tensor

$$F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$$

for the gauge potential A_μ.

(a) (1 point) Name the fundamental fields of the theory.

(b) (1 point) Show that this theory has a global U(1) symmetry which is mediated by

$$\Phi'(x) = \exp(i\Lambda)\Phi(x) \tag{1}$$

where Λ is a constant and thus does not depend on the coordinates.

(c) (2 points) This global U(1) symmetry can be turned into a local symmetry. To this end, promote the parameter Λ from (1) to a function of the space-time coordinates $\Lambda = \Lambda(x)$. In this case, the gauge potential transforms as

$$A'_\mu = A_\mu + \frac{1}{e} \partial_\mu \Lambda \,.$$

Use this transformation and (1) to show how the covariant derivative $D_\mu \Phi$ transforms.

(d) (1 point) Argue form the transformation in the last task that the last two terms of the action are invariant on their own.

(e) (2 points) Show that the kinetic term $F_{\mu\nu} F^{\mu\nu}$ in the action is invariant under the U(1) gauge symmetry, too.

(f) (3 points) Compute the field equations for Φ and A_μ.

(g) (2 points) One of these equations is

$$\partial_\nu F^{\nu\mu} = J^\mu$$

with the current

$$J^\mu = -ie \left(\overline{\Phi} D^\mu \Phi - \Phi \overline{D^\mu \Phi} \right) \,.$$

Show that this current is conserved. *Hint: The actual computation is simple. Just state the definition of a conserved current and then use what you have cleverly.*

(h) (2 points) It is actually very hard to compute solutions for the full field equations. Therefore, we simplify the situation by assuming a very small electrical charge e. In this case, we can proceed by first solving just the standard Klein-Gordon equation

$$\partial_\mu \partial^\mu \Phi + m^2 \Phi = 0\,.$$

Obtain a static solution by using the ansatz

$$\Phi(x,t) = \Psi(x) e^{-imt}$$

and compute the current component J^0 in the leading order of e.

(i) (1 point) Now let us look at the field A_μ. Assume we have a given current $J^\mu = \begin{pmatrix} \rho & 0 & 0 & 0 \end{pmatrix}$, compute the solution for A^μ from the equations of motion by ignoring all contributions from Φ.

2. Conserved currents and the Poincaré algebra

15 points

Consider a real scalar field $\phi(x)$ described by the Klein-Gordon action

$$S = \frac{1}{2}\int \mathrm{d}^4x \left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) .$$

An infinitesimal coordinate transformation acts as

$$x'^\mu = x^\mu + \epsilon^\mu + \omega^\mu{}_\nu x^\nu .$$

Here ϵ^μ is a constant translation vector and $\omega_{\mu\nu}$ (also constant) is an antisymmetric tensor which represents rotations and boots.

(a) (2 points) Using Noether's theorem, derive the conserved current (the energy momentum tensor) $T^\mu{}_\nu$ associated with spacetime translations

$$\phi'(x) = \phi(x-\epsilon) .$$

(b) (2 points) Show that the current you obtained is indeed conserved on-shell.

(c) (2 points) What are the corresponding charge densities that after integration give rise to the conserved charges? Compute them from $T^\mu{}_\nu$.

(d) (1 point) For an infinitesimal Lorentz transformation, the coordinate shift is $\delta x^\mu = \omega^\mu{}_\nu x^\nu$. Write down the field transformation $\delta\phi = \phi(x') - \phi(x)$.

(e) (3 points) Use this transformation to compute the conserved current for the Lorentz symmetry $M^{\mu\nu\rho}$. *Hint: You should find*

$$M^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} .$$

(f) (1 point) Check that $M^{\mu\nu\rho}$ is conserved. *Hint: Use the conservation of the energy momentum tensor you already verified above.*

(g) (2 points) Now analyze at the infinitesimal level what happens when you commute a Lorentz transformation with a translation,

$$\delta_{[\omega,\epsilon]}\phi = \delta_\omega\delta_\epsilon\phi - \delta_\epsilon\delta_\omega\phi .$$

The result is a translation, obtain its parameter.

(h) (2 points) In the last task, you derived parts of the Lie algebra which underlies the Poincaré group. More precisely, you obtained schematically the commutator $[J,P] = P$ where J corresponds to Lorentz transformations and P to translations. What are the other commutators? You do not have to present the full index structure.

3. Spontaneous symmetry breaking

15 points

We take a look at the complex scalar field with the action

$$S = \int \mathrm{d}^4x \left(\overline{\partial_\mu \Phi} \partial^\mu \Phi - V(\Phi)\right) \tag{2}$$

and the potential

$$V(\Phi) = -\mu^2 \overline{\Phi}\Phi + \lambda(\overline{\Phi}\Phi)^2 \,, \qquad \mu^2, \lambda > 0 \,.$$

(a) (2 points) Which values of the field will minimize the potential? Construct from them a solution Φ_0 which is invariant under translations and Lorentz transformations.

(b) (2 points) Expand the action up to quadratic order around this solution. *Hint: You ansatz for the field should look like:*

$$\Phi = (\Phi_0 + \sigma(x))\, e^{i\eta(x)} \,. \tag{3}$$

Use the expanded action to show that

i. (1 point) $\eta(x)$ is a massless field, and

ii. (1 point) $\sigma(x)$ is massive.

(c) (1 point) State the Goldstone theorem and explain how it confirms your results.

The action (2) has a global U(1) symmetry which we now gauge. As result, we obtain again scalar electrodynamics (only this time with a potential):

$$S = \int \mathrm{d}^4x \left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{D_\mu \Phi} D^\mu \Phi - V(\Phi)\right) \,. \tag{4}$$

For details on the field strength tensor and the covariant derivative, refer to the first problem.

(d) (2 points) Write down the expansion ansatz for A_μ. To preserve Lorentz symmetry, expand around $A_\mu = 0$. But this time, apply a gauge transformation to remove the field $\eta(x)$ from the expansion ansatz (3). Remember, we have done the same in the lecture. This special gauge is called unitary gauge.

(e) (2 points) Expand the action (4) up to quadratic order. From the result show that

i. (1 point) a massive scalar (the Higgs boson), and

ii. (1 point) a massive vector field arise.

(f) (2 points) By obtaining a static, point-like solution for the massive vector field's field equations show that it gives rise to a Yukawa potential.

4. Quick questions 15 points

(a) (2 points) Name two important kind of symmetries which appear in classical field theories and provide an example for each.

(b) (1 point) State Noether's theorem.

(c) (2 points) How can massless scalar fields arise from spontaneous symmetry breaking according to Goldstone's theorem?

(d) (2 points) How many generators has the Poincaré group in four dimensions. What kind of transformations do they mediate?

(e) (2 points) What is the defining property of the Lorentz group?

(f) (3 points) Draw a light-cone in two-dimensional Minkowski space (one time and one space direction). Indicate a light-, time- and space-like vector. Which points are causally connected to the observer at the origin on the light-cone?

(g) (2 points) What are the two postulate which naturally lead to the framework of special relativity?

(h) (1 point) How does a non-Abelian symmetry differs from an Abelian one?