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Quantum Field Theory – Practice Exam

15th of June 2023

Please fill in:

Name: ______________________________

Matriculation Number: ______________________________

Number of Sheets: ______________________________

Instructions – Please read carefully:

- Please write your full name and matriculation number on each sheet you hand in.
- Use a separate sheet of paper for each individual problem.
- You have 120 minutes to answer the questions.
- No resources are allowed.
- Use a blue or a black permanent pen.

Do not write below this line.

Comments:

Question:	1	2	Total
Points:	25	25	50
Score:			

In the following, we take a look at scalar QED as it allows to study various of the phenomena and techniques we learned in the course in a quite simple setup. Especially, it does not require to deal with fermions. Instead there is a complex scalar field ϕ which lives in d-dimensional Minkowski spacetime with $g_{\mu\nu} = \mathrm{diag}(1, -1, \dots, -1)$. To make things more interesting, we include a quartic interaction term, resulting in the Lagrangian

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* - \frac{\lambda}{4} (\phi \phi^*)^2 \,. \tag{1}$$

Note that ϕ^* denotes complex conjugation. In the first part of the exam, we discuss some basic properties of this theory and show how it couples to a U(1) gauge field (and therefore, rightfully deserves the name scalar QED). Afterwards, the second part deals with its one-loop renoramlisation.

1. The classical theory 25 points

(a) (2 points) Obtain the equations of motion

$$\partial_\mu \partial^\mu \phi - m^2 \phi - \frac{\lambda}{2} |\phi|^2 \phi = 0$$
$$\partial_\mu \partial^\mu \phi^* - m^2 \phi^* - \frac{\lambda}{2} |\phi|^2 \phi^* = 0$$

for ϕ and its complex conjugate.

(b) (1 point) Show that the theory has a one continuous U(1) symmetry,

$$\phi \to \exp^{ie\lambda} \phi \qquad \text{with} \qquad \lambda \in [0, 2\pi)$$

and e a constant.

(c) (2 points) Derive the Noether current

$$J^\mu = ie \left(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi \right)$$

associated to this symmetry.

(d) (2 points) Verify that the current you found is conserved if the equations of motion are satisfied.

(e) (4 points) Gauge the U(1) symmetry of the theory by introducing the gauge potential A_μ, which transforms infinitesimally as

$$\delta A_\mu = \partial_\mu \lambda \,.$$

Now $\lambda = \lambda(x)$ is coordinate dependent. Use the minimal coupling to couple the scalar field to an electromagnetic field. This implies that you have to add at least a term

$$-j^\mu A_\mu \tag{2}$$

to the original Lagrangian. Write down the full, gauged Lagrangian, including the kinetic term for the gauge field. Verify that it is invariant under the local U(1) symmetry we wanted to gauge.

(f) (3 points) Derive the classical equations of motion for all fields in the gauged theory (including the gauge field). You will find the relativistic form of the inhomogeneous Maxwell equations.

(g) (2 points) Compare the Noether current obtained from the original Lagrangian (before gauging it) to the current appearing on the right hand side of Maxwell's equations that you found now. Which of them is gauge invariant?

(h) (3 points) Sketch the all Feynman diagrams describing interactions of the theory. Which interaction term is new and does not appear in QED with fermionic matter?

For the rest of this problem, we choose the imaginary mass

$$m = i\sqrt{\frac{\lambda}{2}}$$

as result the potential energy of the Lagrangian (1) will look like a Mexican hat. As you might guess, this will result in the Higgs-effect, breaking the U(1) gauge symmetry which we just constructed.

(i) (1 point) Compute the minima of the potential that arises from (1).

(j) (2 points) We now parameterise fluctuations around these minima as

$$\phi = e^{i\xi}(\eta - v)\,,$$

where v is the absolute value of the minima's position you just computed. But what is the physical interpretation of the two new field ξ and η?

(k) (3 points) Finally, plug this ansatz for the field ϕ into the action. Show that by a gauge transformation, one can always eliminate ξ. *(Hint: You should show that ξ only appears in the combination $\partial_\mu \xi + eA_\mu$.)* After this transformation, expand the Lagrangian up to quadratic order in the fields and read off the Proca mass of the gauge field A_μ and the mass of η.

2. Its quantisation and renormalisation

25 points

(a) (4 points) Starting from the Lagrangian (1), explain how to derive the Feynman propagator

$$D_\phi = \lim_{\epsilon \to 0} \frac{i}{p - m^2 + i\epsilon}$$

for the scalar field. Moreover, show how the interaction term (2) gives rise the vertex

$$D_{\phi\phi^* A} = ie(p + p')_\mu \,.$$

Draw the corresponding Feynman diagrams.

(b) (3 points) By taking into account that the photon propagator has the same momentum dependency as the scalar propagator and all three possible interactions you found in h) of the first problem, compute the superficial degree of divergences of the theory in four dimensions. You should get

$$D = 4 - E\,,$$

where E denotes the number of external legs.

(c) (3 points) The theory has a discrete symmetry under complex conjugation, $\phi \to \phi^*$ and $e \to -e$. Therefore, only diagrams with an even number of scalar legs are allowed. Based on this observation draw all six possible divergent diagrams.

(d) (1 point) Which of them is responsible for the renormalisation of the scalar propagator D_ϕ?

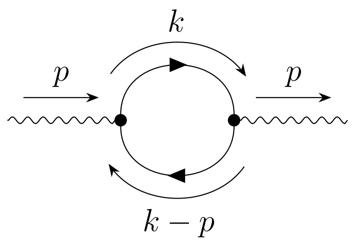
(e) (3 points) How many counterterms are required to cancel its divergences? Write them down and derive their Feynman rules?

(f) (1 point) How does one write the full propagator in terms of the one-particle irreducible (1PI) diagrams?

In the same vein as for the scalar field, we now consider the 1PI diagram for the photo progagator.

(g) (2 points) A contribution to it is given by the Feynman diagram

$$-i\Sigma^{\mu\nu}_{(1)}(p) =$$



Write down the loop integral for this diagram, by using the Feynman rules you obtained in a).

(h) (3 points) To further evaluate this integral, introduce the Feynman parameter x to combine the denominators arising from the propagators in the loop.

(i) (1 point) Explain, why you can remove all terms that are linear in k^μ in the numerator of the results and why you can identify $k^\mu k^\nu$ with $g^{\mu\nu}k^2/d$ under the $\int \mathrm{d}^d k$ integral.

(j) (4 points) Use the results from the two previous task to compute the divergent contribute to $\Sigma^{\mu\nu}_{(1)}(p)$ with dimensional regularisation in $d = 4 - \epsilon$ dimensions.

Hints:

1. *Feynman parameter formula:*

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{(xA + (1-x)B)^2} \, .$$

2. *In $d = 4 - \epsilon$ dimensions*

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^{2a}}{(k^2 + \Delta)^b} = i \frac{1}{(4\pi)^{d/2}} \frac{1}{\Delta^{b-a-d/2}} \frac{\Gamma(a + d/2)\Gamma(b - a - d/2)}{\Gamma(b)\Gamma(d/2)}$$

where $\Gamma(\epsilon) = \epsilon^{-1} + \mathcal{O}(\epsilon)$.