QFT : ϕ^4 notes

- Let us consider the following theory with Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4$$

We will denote the free part $\mathcal{L}_0 \equiv \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2$, and the interaction part as $\mathcal{L}_{int} \equiv -\frac{\lambda}{4!}\phi^4$.

- Essentially, the whole philosophy of perturbative QFT is to study the physical quantities of the full theory $(\mathcal{L}_0 + \mathcal{L}_{int})$ by treating $\mathcal{L}_{int}$ as a small perturbation from the free theory $\mathcal{L}_0$, which we know how to solve. For example, with only $\mathcal{L}_0$ we know how to compute the 2-point function: it is simply the propagator; it is a Green's function, meaning that it is the inverse of the operator $(\partial^2 - m^2)$. In momentum space, ∂_μ becomes p_μ, so essentially we get:

$$\boxed{(p^2 - m^2) D_F(p) = i}$$

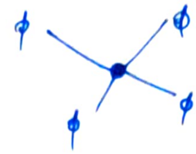
$$\hookrightarrow \boxed{D_F(p) = \frac{i}{p^2 - m^2 + i\varepsilon}}$$

this represents the 2-point function, so diagrammatically, we represent it as

$$= \frac{i}{p^2 - m^2 + i\varepsilon}.$$

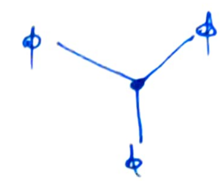
- The interaction part gives the vertex / the vertices;

e.g. (i) ϕ^4 has intuitively 4 scalar fields ϕ at one point:



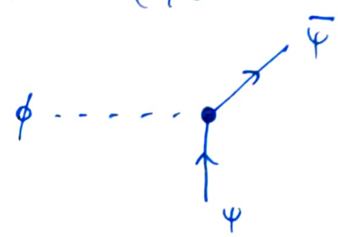
$= -i\lambda$

(ii) $\mathcal{L}_{int} = \frac{-g}{3!}\phi^3$ has



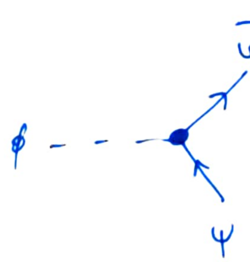
$= -ig$

(iii) $\mathcal{L}_{int} = -g\phi\bar{\psi}\psi$ (Yukawa theory) has the vertex



$= -ig$

(iv) $\mathcal{L}_{int} = -ig\bar{\psi}\gamma^5\psi\phi$ (pseudoscalar Yukawa) has the vertex



$= g\gamma^5$,

and the pattern continues for other interactions. An exercise for you: what would an interaction vertex $\mathcal{L}_{int} = \frac{-\lambda}{6!}\phi^6$ look like? And what would be the Feynman rule?

N.B. QED is essentially Yukawa theory, but we replace scalar ϕ by EM potential A_μ, giving $\mathcal{L}_{int} = -e\bar{\psi}\gamma^\mu\psi A_\mu$. What would be the Feynman rule?

QFT : ϕ^4 notes

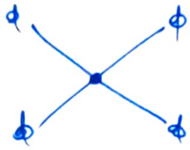
- To summarise, from

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2}_{\mathcal{L}_0} \underbrace{- \frac{\lambda}{4!}\phi^4}_{\mathcal{L}_{int}} ,$$

we get from $\mathcal{L}_0$:

$$= \frac{i}{p^2 - m^2} ,$$

and from $\mathcal{L}_{int}$:

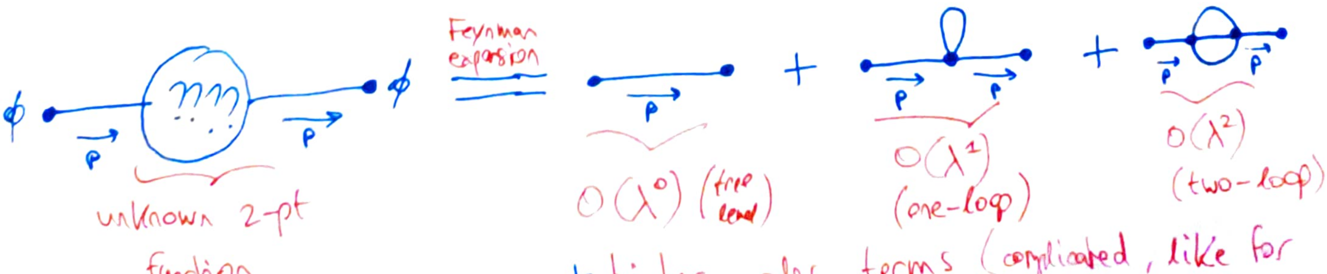


$$= -i\lambda .$$

These are essentially the <u>2-point and 4-point functions</u> at <u>zeroth order</u> in the perturbative expansion in λ.

The  $= \frac{i}{p^2-m^2}$ is the 2-point function for $\mathcal{L}_0$, but NOT for the <u>full, interacting</u> theory $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$!

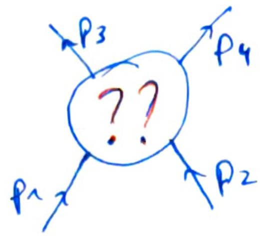
→ One can obtain the 2-point function for $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$ by <u>perturbative expansion</u>, using <u>Feynman diagrams</u> and the <u>Feynman rules</u> ! Schematically, we have the expansion :



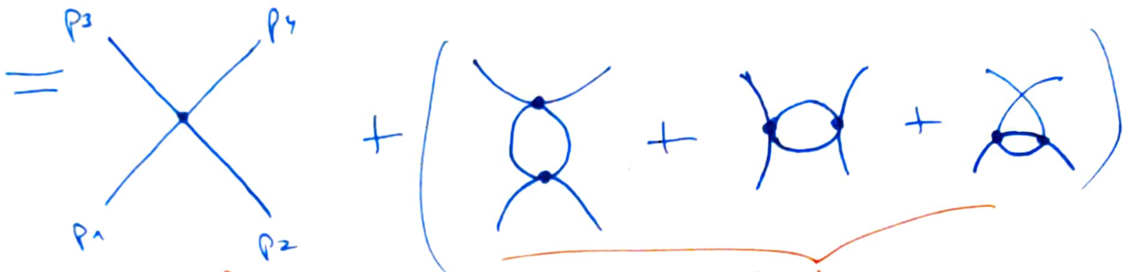
+ higher order terms (complicated, like for example , etc.)
...

• What about the 4-point function? $i\mathcal{M}(p_1 p_2 \to p_3 p_4)$

$i\mathcal{M}(p_1 p_2 \to p_3 p_4) =$



(unknown, but can find it by perturbation)



+ higher order terms

↓

more complicated diagrams

$O(\lambda^1)$ (tree level)

$O(\lambda^2)$ (one-loop)

• Physically, this 4-point function allows to compute the scattering amplitude of e.g. the process $(\phi\phi \to \phi\phi)$, through

$$S = \sum |\mathcal{M}|^2 .$$

↳ this is what is measured in experiments! (or rather, cross sections)

• What about divergences? these can happen when computing certain loop diagrams... Yet because the scattering amplitude is a physical quantity, it should not be $= \infty$ (unphysical!)!

→ Schematically, a loop gives integration $\sim \int d^d k$, while an internal propagator gives in the integral $\sim \frac{1}{k^2}$

(by direct application of Feynman rules).

QFT : ϕ^4 notes

Superficial degree of divergence in ϕ^4

↳ one would like to check before actually calculating, whether a diagram is divergent or NOT.

↳ if
- L = number of loops in a diagram,
- P = number of propagators (internal lines), and
- N = number of external lines, and
- V = number of vertices.

- number of loops in a diagram is not independent :

$$\boxed{L = P - V + 1}$$

- There are 4 lines meeting at each vertex in ϕ^4 (),

so
$$\boxed{4V = N + 2P}$$

(Exercise: what is the relation for ϕ^n vertex ?),

- Superficial degree of divergence of a diagram (D) is thus

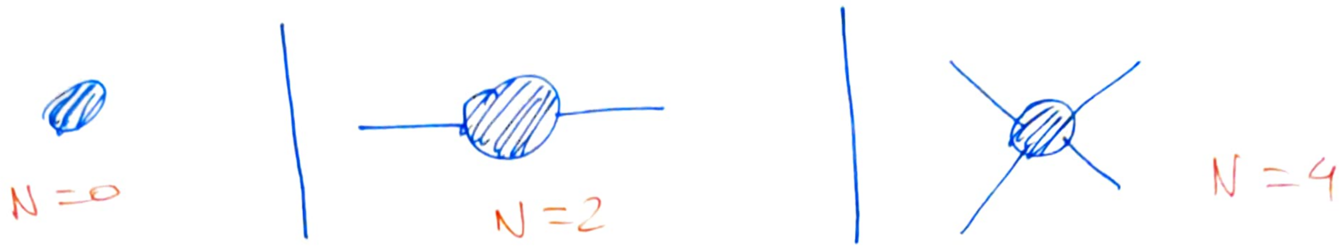
$$D = \underbrace{dL}_{\left(\int d^d k\right)^L} - \underbrace{2P}_{\left(\frac{1}{k^2}\right)^P}$$

(# k's in numerator minus # k's in denominator)

$$= d + \big(2(d-2) - d\big)V - \frac{(d-2)}{2}N .$$

↳ $\boxed{D \overset{d=4}{=} 4 - N}$ for ϕ^4 in $d = 4$!

- due to $\phi \to -\phi$ symmetry, diagrams with odd number of external legs vanish. (so no $N=1$, $N=3$)
- so $\boxed{D \geq 0}$ (divergence) if and only if $\boxed{N = 0, 2, 4}$



- in ϕ^4, these are the diagrams which naively diverge. (other diagrams are perfectly fine, because $D < 0$.)
- "bubble" diagram drops out of physical quantities; it is an unobservable shift of the vacuum energy.

- **Problem**:

Remember, we wanted to compute the 2-point and 4-functions of the full theory $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$.

→ Well, this analysis tells us that the corrections to the 2 and 4-point functions actually diverge!

↳ we need to regularise and renormalise these diagrams in order to obtain finite results (physical)!

QFT: ϕ^4 notes

- Having (i) found the Feynman rules for $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$
  (ii) identified the divergent diagrams in our theory,

we are now ready to renormalise our theory. Instead of explaining the philosophy behind renormalisation, I will simply describe the procedure (for details, see lectures + tutorials).

(1) rescale field as $\boxed{\phi \longrightarrow \sqrt{Z_\phi}\,\phi}$. Our Lagrangian is now,

$$\mathcal{L} = \frac{1}{2} Z_\phi (\partial_\mu \phi)^2 - \frac{m^2}{2} Z_\phi \phi^2 - \frac{\lambda}{4!} Z_\phi^2 \phi^4 .$$

(2) rescale also m^2 and λ such that it cancels out the Z_ϕ's in the terms with m^2 and λ. This means

$$m^2 Z_\phi \longrightarrow m^2 Z_m \quad \text{and} \quad \lambda Z_\phi^2 \longrightarrow \lambda Z_\lambda ,$$

(corresponds to $m^2 = \frac{Z_m}{Z_\phi} m_R^2$ and $\lambda = \frac{Z_\lambda}{Z_\phi^2} \lambda_R$, but we drop subscript R)

Our Lagrangian now becomes,

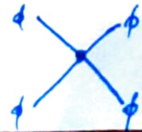
$$\boxed{\mathcal{L} = \frac{1}{2} Z_\phi (\partial_\mu \phi)^2 - Z_m \left(\frac{m^2}{2} \phi^2\right) - Z_\lambda \left(\frac{\lambda}{4!} \phi^4\right)}.$$

(each term has its own Z_i)

↳ convince yourself that this just looks like "ordinary" ϕ^4, with the modified Feynman rules:

$$= \frac{i}{Z_\phi p^2 - Z_m m^2}$$

and



$$= -i Z_\lambda \lambda .$$

- At this point, we could already start computing diagrams and absorbing divergences. BUT, a better way to do things is to split the Lagrangian, by eliminating the Z_i's through

$$\begin{cases} Z_\phi := 1 + \underbrace{\delta_{Z_\phi}} \\ Z_m := 1 + \underbrace{\delta_{Z_m}} \\ Z_\lambda := 1 + \underbrace{\delta_{Z_\lambda}} \end{cases} \quad \text{(counterterms)}.$$

- Our Lagrangian now becomes,

$$\mathcal{L}_{TOT} = \left.\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \quad \right] \text{regular } \phi^4 \text{ Lagrangian } \mathcal{L}_{\phi^4}$$

$$\underbrace{+ \frac{1}{2}\delta_{Z_\phi}(\partial_\mu\phi)^2 - \delta_{Z_m}\left(\frac{m^2}{2}\phi^2\right) - \delta_{Z_\lambda}\left(\frac{\lambda}{4!}\phi^4\right)}_{\mathcal{L}_{c.t.} \text{ (counterterm Lagrangian)}}.$$

- Remember, this is simply a rewriting of the previous Lagrangian with the Z_i's, just to better organise the perturbation theory!

Our old Feynman rules become, simply by substitution:

$$= \frac{i}{p^2 - m^2 + (\delta_{Z_\phi}p^2 - \delta_{Z_m}m^2)}, \qquad (1)$$

$$= -i\lambda - i\,\delta_{Z_\lambda}\lambda . \qquad (2)$$

QFT: ϕ^4 notes

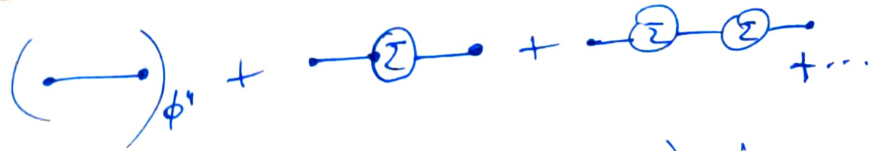
- Let us look at equation (1) for the (renormalized) propagator.

$$\left(\overset{p}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet}\right)_R = \frac{i}{p^2-m^2+\underbrace{\left(\delta_{Z_\phi}p^2-\delta_{Z_m}m^2\right)}} \equiv \frac{i}{p^2-m^2+\Sigma(p^2)}$$

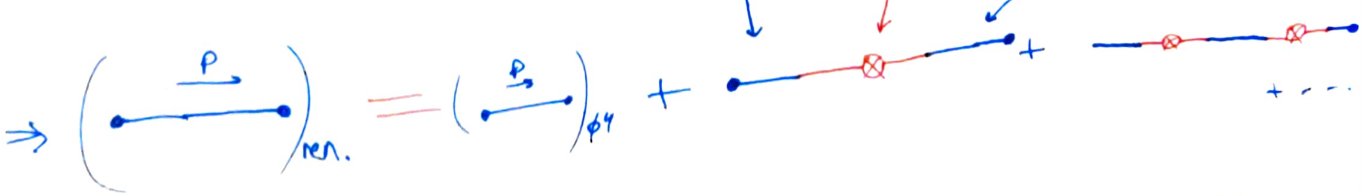
- By treating the δZ_i's as small parameters, we can Taylor expand (geometric series $\frac{1}{1+x} \approx 1-x$) the propagator (remember Problem Sheet 5 !!!). This gives,

$$\left(\overset{p}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet}\right)_{Ren.} = \underbrace{\frac{i}{p^2-m^2}}_{\text{ordinary } \phi^4 \text{ propagator}} + \frac{i}{p^2-m^2}\, i\Sigma(p^2)\, \frac{i}{p^2-m^2} + \ldots$$

interpretation

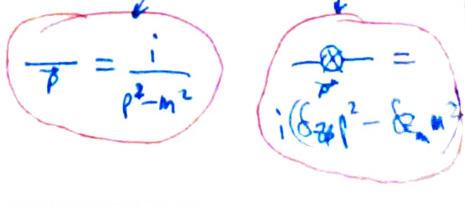


$$= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\left(i\delta_{Z_\phi}p^2 - i\delta_{Z_m}m^2\right)\frac{i}{p^2-m^2} + \ldots$$

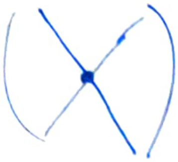
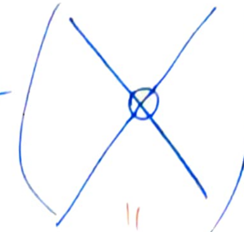


→ FROM this expansion of our "old" propagator, one obtains the Feynman rules of our "new" theory $\mathcal{L}_{tot} = \mathcal{L}_{\phi^4} + \mathcal{L}_{c.t.}$

(remember: just a rewriting)



- equation (2) just tells us that

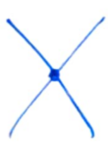
 $_{\text{renormalised}}$ $=$  $_{\text{ordinary } \phi^4}$ $+$  $_{\text{counterterm}}$,

giving the Feynman rules $= -i\lambda$ $\qquad -i\,\delta_{Z_\lambda}\lambda$.

- To summarise, with our Lagrangian $\mathcal{L}_{tot} = \mathcal{L}_{\phi^4} + \mathcal{L}_{c.t.}$, we get the following Feynman rules:

$= \dfrac{i}{p^2 - m^2}$;

$= i\left(\delta_{Z_\phi} p^2 - \delta_{Z_m} m^2\right)$

 $= -i\lambda$;  $= -i\,\delta_{Z_\lambda}\lambda$,

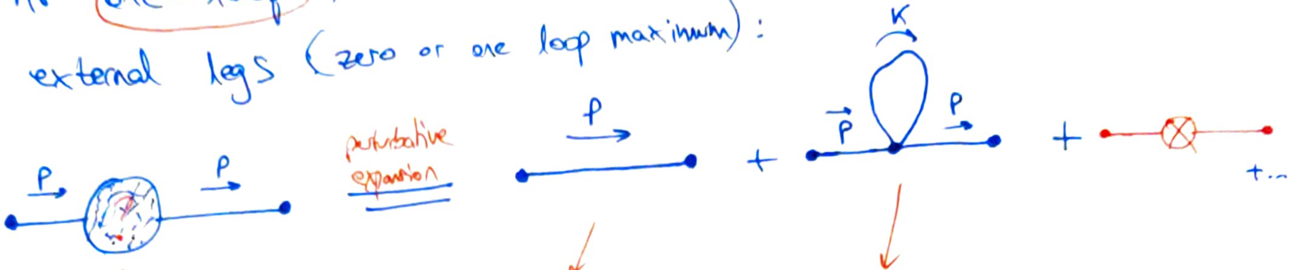
along with all the usual Feynman rules (integrate over loop momenta, include symmetry factors, etc.)

- We are now ready to compute the 2 superficially divergent diagrams  and  (at one-loop).

- We will absorb the divergences of these diagrams in the counterterms, which will leave us with finite values for the 2- and 4-point functions, allowing to find the physical (and measurable) scattering amplitudes and cross sections.

(I) 2-point function

. At one-loop, draw all possible diagrams which have two external legs (zero or one loop maximum):



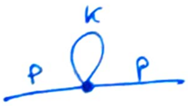
(what happens between the 2-points) = (nothing special) + (loop at a point in between the 2 points) + (counterterm diagram)

+ higher order terms ($O(\lambda^2)$ and higher; here we only look at $O(\lambda)$)

↳ the first diagram we know already: this is just the free propagator $\frac{i}{p^2-m^2}$.

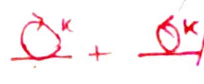
↳ the second diagram is a loop diagram, which is the one we are interested with, and which potentially diverges.

↳ We apply the Feynman rules to this diagram:



$$= \overset{\text{vertex}}{(-i\lambda)} \cdot \overset{\text{sym. factor}}{\frac{1}{2}} \cdot \int \overset{\text{undetermined momentum}}{\frac{d^dK}{(2\pi)^d}} \overset{\text{internal propagator}}{\frac{i}{K^2-m^2}}$$

(! Symmetry factor! of 1/2



)

$$= \frac{-i\lambda}{2} \cdot \frac{1}{(4\pi)^{d/2}} \cdot \frac{\Gamma(1-d/2)}{(m^2)^{1-d/2}}$$

(Formula given in appendix of Peskin & Schroeder; will be given in practice & exam!)

- now we do <u>dimensional regularisation</u> : let $\boxed{d = 4 - \varepsilon}$,

where $\varepsilon \ll 1$. We expand the result in ε to identify the divergent term (typically called the 'pole').

- Use Taylor expansion $a^{\varepsilon} = 1 + \varepsilon \ln a + O(\varepsilon^2)$

and expansion of Gamma function $\Gamma(\varepsilon) \approx \frac{1}{\varepsilon} - \gamma + o(\varepsilon)$.

Also need fundamental property of Gamma function:

$$\Gamma(\varepsilon) = (\varepsilon - 1) \cdot \Gamma(\varepsilon - 1)$$, and expansion

$$\frac{1}{\varepsilon - 1} \approx -1 - \varepsilon .$$

- This all gives the following expansions in the previous formula,

$$\frac{1}{(4\pi)^{d/2}} \approx \frac{1}{(4\pi)^2}\left(1 + \frac{\varepsilon}{2}\ln(4\pi)\right)$$

$$\left(m^2\right)^{d/2 - 1} \approx \frac{1}{m^2}\left(1 + \frac{\varepsilon}{2}\ln(m^2)\right)$$

$$\Gamma(1 - d/2) \approx \frac{\Gamma(\varepsilon/2)}{(\varepsilon/2 - 1)} \approx \Gamma(\varepsilon/2)\left(-1 - \varepsilon/2\right)$$

$$\approx -\frac{2}{\varepsilon} + \gamma - 1 ,$$

all combining to give for the diagram, looking only at pole,

 $\approx \frac{i\lambda}{(4\pi)^2 m^2} \cdot \frac{1}{\varepsilon}$ (+ non-divergent terms as $\varepsilon \to 0$)

- Remember, full 2-point function is :

$$\simeq \frac{i}{p^2-m^2} + \frac{i\lambda}{(4\pi)^2 m^2} \cdot \frac{1}{\varepsilon} + i\left(\delta_{Z_\phi} p^2 - \delta_{Z_m} m^2\right)$$

$$+ \left(\text{regular terms as } \varepsilon \to 0\right) .$$

→ we select the counter-terms such that they cancel out the divergence $\left(\sim \frac{1}{\varepsilon}\right)$. This means,

$$\boxed{\begin{aligned} \delta_{Z_\phi} &\overset{!}{=} 0 \\ \delta_{Z_m} &\overset{!}{=} \frac{\lambda}{(4\pi)^2 m^4} \cdot \frac{1}{\varepsilon} \end{aligned}} .$$

This leaves us with the (physical) 2-point function at one-loop :

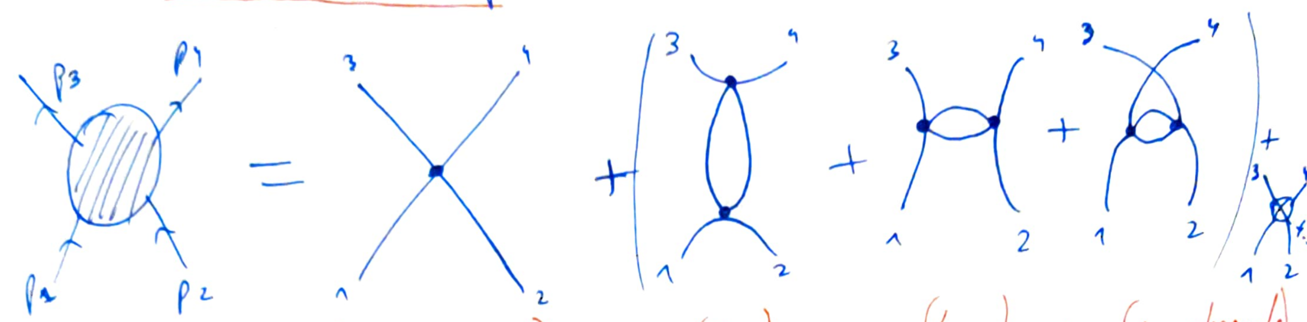
$\simeq \frac{i}{p^2-m^2}$ + (regular terms as $\varepsilon \to 0$, which we can calculate if we want, but don't need for counterterm)

now, NOT infinite! FINITE!

→ δ_{Z_m} appears in Lagrangian and contains the divergence; but Lagrangian is NOT an observable ... so "OK".

(2) Four-point function @ one-loop:

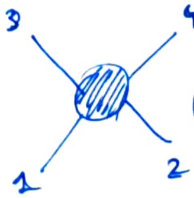
Draw all possible diagrams for the interaction ($\phi\phi \to \phi\phi$) ($p_1 p_2 \to p_3 p_4$) with maximum one-loop:



(what happens $p_1 p_2 \to p_3 p_4$?) = ("nothing" happens) + (s-) + (t-) + (u-channels)

+ (counterterm) + (higher-order terms)

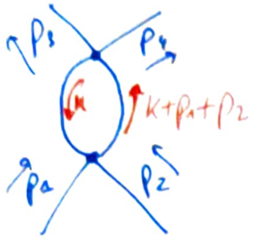
↳ first diagram we already know! "free" 4-point function of ϕ^4:  $= -i\lambda$.

↳ Remember: we expect a divergence in the full 4-point function. The counterterm  is going to cancel the poles, leaving us with a finite  (total 4-pt Fct).

- Roughly: for scalars, the 3 s, t, and u-channels are essentially given by the same function, say $V(p^2)$.

→ if $V(p^2)$ has a pole $\sim 1/\varepsilon$, then  needs to be selected to be $\sim (-3) \cdot 1/\varepsilon$, to cancel the poles of each of the 3 diagrams! Let us do this in detail.

- Let us look at the s-channel:



$$\xrightarrow{\text{Feynman rules}} \underbrace{(-i\lambda)}_{1^{st}\text{ vertex}}\underbrace{(-i\lambda)}_{2^{nd}\text{ vertex}}\cdot\underbrace{\left(\frac{1}{2}\right)}_{\text{sym. factor}}\underbrace{\int\frac{d^dk}{(2\pi)^d}}_{\text{integrate over undetermined momenta}}\underbrace{\frac{i}{(k^2-m^2)}\frac{i}{((k+p_1+p_2)^2-m^2)}}_{\text{2 internal propagators between the 2 vertices}}$$

$$=\frac{\lambda^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2)}\cdot\frac{1}{((k+p)^2-m^2)}$$

(I renamed $p_1+p_2\equiv p$)

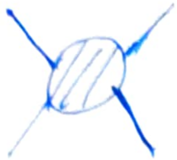
↳ Interlude: at this point, the strategy is to use Feynman parameters $\frac{1}{AB}=\int\dots$, complete the square $l=k+xp$, and then use formulas for loop integrals (or Wick rotate first, etc.)

(!) Trick if you just need to compute divergent part of the integral (so not including all the finite, regular terms):

→ Set all external momenta to zero:

$p_1=p_2=p_3=p_4=0$; the divergence only comes from loop, which doesn't care about the values of the p_i's. This simplifies calculations if you only need the pole for the counterterm (!)

("minimal substraction scheme")



$$\sim \frac{\lambda^2}{2}\int\frac{d^dK}{(2\pi)^d}\,\frac{1}{(K^2-m^2)^2}$$

formulas appendix / exam
(n=2) (A.44)

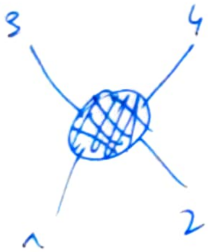
$$= \frac{\lambda^2}{2}\,\frac{i}{(4\pi)^{d/2}}\,\frac{\Gamma(2-d/2)}{(m^2)^{2-d/2}}$$

$d = 4-\varepsilon$ and expand, with all usual formulas.
(only need pole from $\Gamma(\varepsilon/2)$)

$$\boxed{\sim \frac{i\lambda^2}{(4\pi)^2}\cdot\frac{1}{\varepsilon} \text{ + regular terms as } \varepsilon\to 0.}$$

- For s-channel, this is our result.
  We will get exactly same pole for both t- and u-channels.

So, roughly, the 4-point function is something like:



$$\simeq -i\lambda + \underset{(s,t,u)}{3}\cdot\left(\frac{i\lambda^2}{(4\pi)^2}\cdot\frac{1}{\varepsilon}\right) - i\,\delta_{Z_\lambda}\lambda \quad (+\ldots)$$

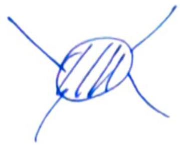
(+ ...)

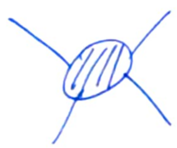
→ divergence cancels out if counterterm is chosen to be

$$\boxed{\delta_{Z_\lambda} \overset{!}{=} \frac{3\lambda}{(4\pi)^2}\cdot\frac{1}{\varepsilon}}\,.$$

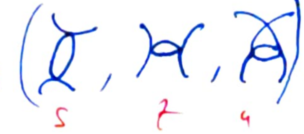
QFT : ϕ^4 notes

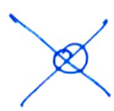
To summarise : in ϕ^4-theory :

- We had 2 superficially divergent diagrams, namely  and . To absorb these divergences, we included  $= i(\delta_{Z_\phi} p^2 - \delta_{Z_m} m^2)$ and  $= -i\,\delta_{Z_\lambda} \lambda$ as counterterms in the Lagrangian.

- AT ONE LOOP, we computed  and  by perturbative expansion.

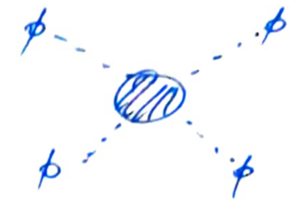
We found that for , the divergence came from , which we absorbed into .

For , the divergence came from .

These 3 diagrams are all determined ~~with~~ by one function, so we only compute one of them to find its pole. All 3 poles are then absorbed into . The counterterms @ 1 loop are:

$$\boxed{\delta_{Z_\phi} = 0 \ ; \ \delta_{Z_m} \simeq \frac{\lambda}{(4\pi)^2 m^4} \cdot \frac{1}{\varepsilon} \ ; \ \delta_{Z_\lambda} \simeq \frac{3\lambda}{(4\pi)^2} \cdot \frac{1}{\varepsilon}}$$

- This procedure depends on each theory. For example, Yukawa theory ($\mathcal{L}_{int} = -g\bar{\psi}\psi\phi$) from Problem 9, one of the superficially divergent diagrams is



$\rightarrow$ To cancel out this divergence, one actually needs to include a ϕ^4 - counterterm to the Lagrangian of Yukawa, (which does not contain a ϕ^4-interaction originally!).

This was the goal of Problem Sheet 9, ex. (f) and (g).