5.2. Example $sl(3,\mathbb{C})$

interesting for physics because complexification of $SU(3)$ ~ gauge group of QCD.

generators are traceless, real 3×3 matrices ; in total

$$\underbrace{2 \text{ diagonal}}_{\text{Cartan subalgebra}} + \underbrace{3 \text{ upper triangular} + 3 \text{ lower tri}^\circ}_{\text{non-zero roots}} = 8$$

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Killing form: $K(X,Y) = \mathrm{Tr}(X\cdot Y)$

We take $c_\alpha = 1$ and therefore $(H_i, H_j) = \begin{pmatrix} \boxed{\begin{matrix} 2 \\ -1 \end{matrix}} & \boxed{\begin{matrix} -1 \\ 2 \end{matrix}} \end{pmatrix}$

taking $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

we find:

$$[H_1, E_1] = \boxed{2}\, E_1 \qquad [H_1, E_2] = \boxed{-1}\, E_2$$
$$[H_2, E_1] = \boxed{-1}\, E_1 \qquad [H_2, E_2] = \boxed{2}\, E_2$$

$E_{-1} = E_1^T$, $E_{-2} = E_2^T$ check $K(E_\alpha, E_{-\beta}) = \delta_{\alpha,\beta}$

finally $E_3 = [E_1, E_2] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $E_{-3} = E_3^T$

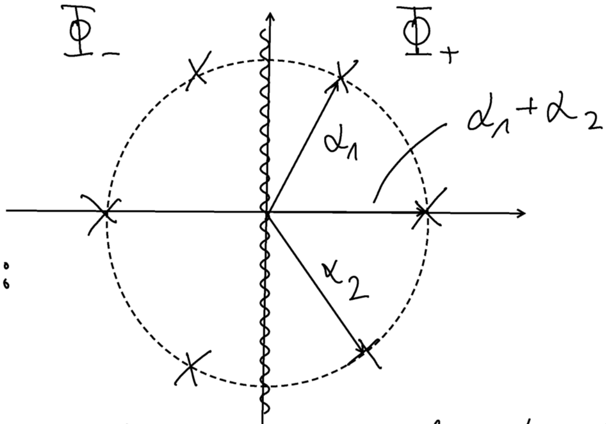
<u>Homework</u> check all the other relations

6. Simple Roots and Cartan Matrices

6.1. Simple Roots

last lecture: We introduced the root system of a simple Lie algebra.

i.e. $su(3)$ (here in adapted basis for the Cartan generators):



Question: What is the minimal set of roots we need to find all the others?

Answer: Simple roots!

(I) find a hyperplane, with no roots on it, that divides the rootsystem into two half spaces

$V_+ \leftarrow$ positive roots

$V_- \leftarrow$ negative roots.

For $\alpha \in V_\pm$, we write $\alpha \gtrless 0$, and split the root system Φ into $\Phi_\pm = \{ \alpha \in \Phi \mid \alpha \gtrless 0 \}$.

Note: $\alpha \in \Phi_+ \Leftrightarrow -\alpha \in \Phi_-$ and $E_\alpha \hat{=}$ raising operator, $E_{-\alpha} \hat{=}$ lowering operator for $\alpha > 0$

number of elements: $\#(\Phi_\pm) = \dfrac{\dim(g) - \mathrm{rank}(g)}{2} \in \mathbb{N}$

triangular decomposition of g:

$$g = g_+ \oplus g_0 \oplus g_- \,, \quad g_\pm = \mathrm{span}_{\mathbb{C}} \{ E_\alpha \mid \alpha \gtrless 0 \}$$

(II) simple roots = all positive roots that cannot be written as of two or more positive roots

Independently of the choice of hypersurface there are $r = \text{rank}\, g$ simple roots. They form a basis of Φ:

$$\text{span}_{\mathbb{R}}\, \Phi_S = \text{span}_{\mathbb{R}}\, \Phi \quad \text{with} \quad \Phi_S = \{\alpha_i \mid i = 1, \dots, r\}$$

any pos. root = sum of simple roots with non-neg. coefficients

remember Cartan-Weyl basis:

$$[H_i, E_\alpha] = \underset{\substack{\shortparallel \\ \text{vector}}}{\alpha_i} E_\alpha \,, \quad [E_\alpha, E_{-\alpha}] = H_\alpha = \alpha^{\vee i} H_i$$

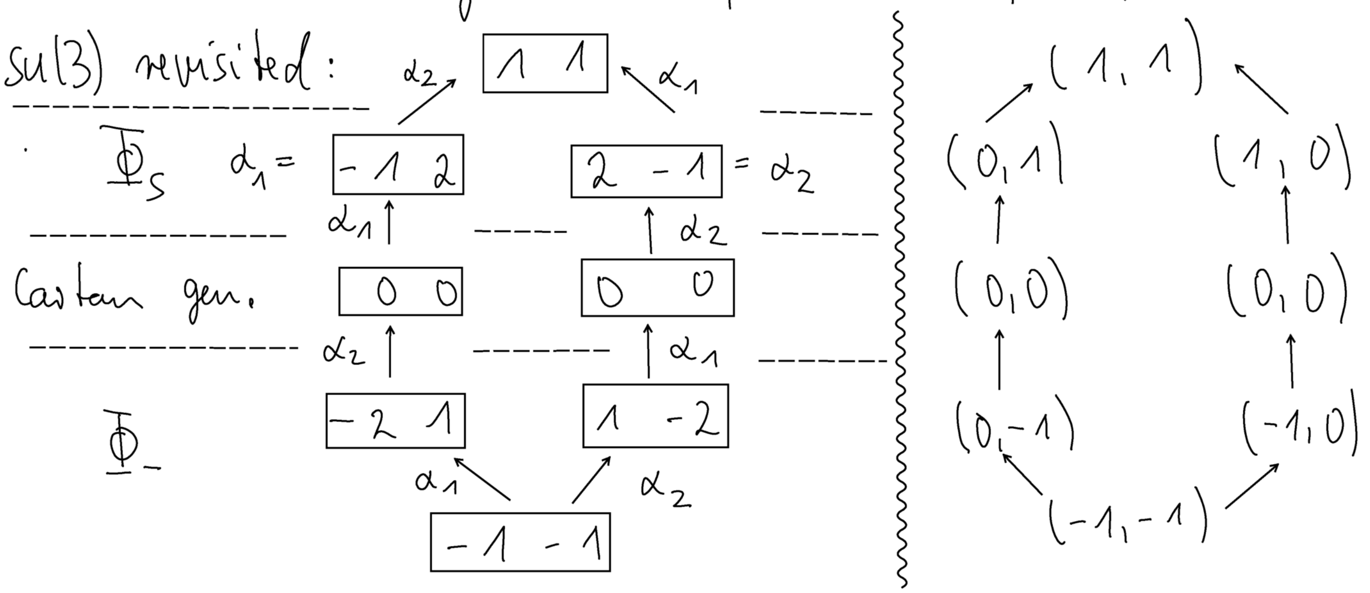
one-form $\hat{=}$ coroot

We can always choose $(\alpha_i)^{\vee j} = \delta_i^j$

such that $H_{\alpha_i} = H_i$.

For an arbitrary coroot $\beta^\vee$ we have $\beta^\vee = \beta^i (\alpha_i)^\vee$.

su(3) revisited:



root basis = Dynkin basis $\qquad$ co-root basis

Lie ART: RootSystem[algebra] $\qquad$ AlphaBasis[root]

i.e. SU3

Question: Relation between root & coroot basis ?
Answer: (inverse) Cartan matrix.

6.2. Cartan matrix

Cartan matrix encodes non-orthogonality of the simple roots

$$A_{ij} := (\alpha_i, \alpha_j^\vee) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \qquad i,j = 1, \dots, \operatorname{rank} g$$

Note that:
- rows of A_{ij} are the components of simple roots in Dynkin basis
- NOT always symmetric
- entries are integers

and moreover:

1) $A_{ii} = 2$: follows from definition

2) $A_{ij} = 0 \Rightarrow A_{ji} = 0$: $A_{ij} = 0$ implies $(\alpha_i, \alpha_j) = 0$ because $(\alpha_i, \alpha_i) \neq 0$ and therefore $A_{ji} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = 0$

3) $A_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$

4) The Cartan matrix is non-degenerate.
simple roots form a basis of root space.

5) $A_{ij} A_{ji} \leq 4$ (no sum):

We introduce the angle $\measuredangle(\alpha, \beta)$ between $\alpha, \beta \in \Phi$

$$\cos \measuredangle(\alpha, \beta) = \frac{(\alpha, \beta)}{\sqrt{(\alpha, \alpha)(\beta, \beta)}}$$

(remember scalar product in LAG)

$$A_{ij} A_{ji} = 4 \cos^2 \measuredangle(\alpha_i, \alpha_j) \leq 4$$

Consequence: only 4 possibilities for $A_{ij}, A_{ji}, i \neq j$:

a) $A_{ij} = A_{ji} = 0$ c) $A_{ij} = -2$, $A_{ji} = -1$

b) $A_{ij} = A_{ji} = -1$ d) $A_{ij} = -3$, $A_{ji} = -1$.

for $A_{ij} A_{ji} = 4$ $\measuredangle(\alpha_i, \alpha_j) = 0 \Rightarrow \alpha_i \parallel \alpha_j$
contradicts that α_i and α_j are linear independent.

Block diagonal structure $A = \begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix}$ is still possible. But then g has proper sub ideals and is semisimple. For simple Lie algebras A is indecomposible.

6.3. Chevalley - Serre Relations

We can now give very compact expressions for the commutators of g in terms of $\{E_{\pm i}, H_i\}$ with $E_{\pm i} = E_{\pm \alpha_i}$. These 3 rank g generators satisfy:

$$\boxed{\begin{aligned} &1)\ [H_i, H_j] = 0 \\ &2)\ [H_i, E_{\pm j}] = \pm A_{ij} E_{\pm j} \\ &3)\ [E_{+i}, E_{-j}] = \delta_{ij} H_j \\ &4)\ (\mathrm{ad}_{E_{\pm i}})^{1-A_{ij}} E_{\pm j} = 0 \end{aligned}}$$

Chevalley - Serre relations

$\hat{=}$ How often can I add α_i to α_j before I leave the root system?