

Last lecture: We encountered the path integral.  
 In particular, we related it to the Feynman propagator

$$\langle 0 | T \hat{\phi}(x_1) \hat{\phi}(x_2) | 0 \rangle = \frac{\int \mathcal{D}\phi \phi(x_1) \phi(x_2) \exp\left[i \int_{-\infty}^{\infty} d^4x \mathcal{L}\right]}{\int \mathcal{D}\phi \exp\left[i \int_{-\infty}^{\infty} d^4x \mathcal{L}\right]} = \frac{I_2}{I_1}$$

required to normalise }  
 vacuum state  $\langle 0 | 0 \rangle = 1$

### 5.3. Wick's Theorem and Feynman Diagrams

Idea: compute RHS directly for  $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$   
 like last lecture discretise in box

$$\phi(x_i) = \frac{1}{L^4} \sum_n e^{-i k_n x_i} \phi(k_n), \quad k_n^M = \frac{2\pi n^M}{L}$$

$$|n^M| < \frac{\pi}{\epsilon}$$

$\phi(x)$  is real  $\Rightarrow \phi^*(k) = \phi(-k)$  or  $\text{Re}/\text{Im}[\phi(k_n)]$   
 as independent variables for  $k_n^0 > 0$

$$\mathcal{D}\phi(x) = \prod_{k_n^0 > 0} d \text{Re} \phi(k_n) d \text{Im} \phi(k_n), \text{ and}$$

$$S = \int d^4x \mathcal{L} = \lim_{\substack{L \rightarrow \infty \\ \epsilon \rightarrow 0}} \left[ \frac{1}{L^4} \sum_{k_n^0 > 0} (m^2 - k_n^2) |\phi_n|^2 \right] \quad \phi_n = \phi(k_n)$$

$$I_1 = \int \mathcal{D}\phi e^{iS} = \prod_{k_n^0 > 0} \int d \text{Re}(\phi_n) \exp\left[\frac{i}{L^4} (m^2 - k_n^2) \text{Re}^2 \phi_n\right]$$

$$\times \int d \text{Im}(\phi_n) \exp\left[\frac{i}{L^4} (m^2 - k_n^2) \text{Im}^2 \phi_n\right]$$

$$= \prod_{\text{all } k_n} \sqrt{\frac{-i\pi L^4}{m^2 - k_n^2}} \quad \text{after } L \rightarrow \infty \text{ \& } \epsilon \rightarrow 0$$

↑ Gaussian integral, see EX.

$$\text{in particular } \left( \prod_k \int d\xi^k \right) \exp \left[ - \xi^i A_{ij} \xi^j \right]$$

$$= \prod_i \left( \int dx^i \exp \left[ -a_i (x^i)^2 \right] \right) = \prod_i \sqrt{\frac{\pi}{a_i}}$$

eigenvectors of  $A$  with eigenvalue  $a_i$   
 $= \text{const} \times [\det A]^{-1/2}$

$$S = \frac{1}{2} \int d^4 x \phi (-\partial^2 - m^2) \phi \text{ after i. b. p., thus}$$

$$\int \mathcal{D}\phi e^{iS} = \text{const} \times [\det (m^2 + \partial^2)]^{-1/2}$$

For  $I_2$  we need:

$$\phi(x_1) \phi(x_2) = \frac{1}{L^8} \sum_{m, l} e^{-i(k_m x_1 + k_l x_2)} \phi_m \phi_l$$

$$I_2 = \dots = \frac{1}{L^8} \sum_m e^{-ik_m(x_1 - x_2)} \underbrace{\left( \prod_{k_n^0 > 0} \frac{-i\pi L^4}{m^2 - k_n^2} \right)}_{I_1} \frac{-i L^4}{m^2 - k_m^2 - i\epsilon}$$

Therefore we obtain after the continuum limit  $L \rightarrow \infty$ ,  
 finally the expected result:  $\epsilon \rightarrow 0$

$$\langle 0 | T \hat{\phi}(x_1) \hat{\phi}(x_2) | 0 \rangle = \lim_{\epsilon \rightarrow 0^+} \int \frac{d^4 k}{(2\pi)^4} \frac{i e^{-ik(x_1 - x_2)}}{k^2 - m^2 + i\epsilon}$$

$$= D_F(x_1 - x_2) //$$

We can now also compute more complicated correlations like

cumbersome, use trick?

$$\langle 0 | T \hat{\phi}(x_1) \hat{\phi}(x_2) \hat{\phi}(x_3) \hat{\phi}(x_4) | 0 \rangle = \dots = \text{Wick contraction}$$

$$\underbrace{\phi_1 \phi_2}_{\text{contract}} \underbrace{\phi_3 \phi_4}_{\text{contract}} + \underbrace{\phi_1 \phi_2 \phi_3}_{\text{contract}} \phi_4 + \underbrace{\phi_1 \phi_2 \phi_4}_{\text{contract}} \phi_3$$

Wick's Theorem: Add all possible contractions with the basic contraction:

$$\phi_i := \phi(x_i) \rightsquigarrow \underbrace{\phi_1 \phi_2}_{\text{contraction}} = \mathcal{D}_F(x_1 - x_2)$$

$$= \mathcal{D}_F(x_1 - x_2) \mathcal{D}_F(x_3 - x_4) + \mathcal{D}_F(x_1 - x_3) \mathcal{D}_F(x_2 - x_4) + \mathcal{D}_F(x_1 - x_4) \mathcal{D}_F(x_2 - x_3)$$

$$= \begin{array}{c} x_1 \\ | \\ x_2 \end{array} \begin{array}{c} x_3 \\ | \\ x_4 \end{array} + \begin{array}{c} x_1 \text{---} x_3 \\ x_2 \text{---} x_4 \end{array} + \begin{array}{c} x_1 \text{---} x_4 \\ x_2 \text{---} x_3 \end{array}$$

These diagrams are called Feynman diagrams.

### 5.4. Interacting Theories

Things became even more interesting with interacting Lagrangians:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} \quad \text{i.e.} \quad \mathcal{L}_{int} = -\frac{\lambda}{4!} \phi^4$$

quadratic  $\rightarrow$  solvable by Gaussian integrals

For small  $\lambda$  we can expand:

$$\exp \left[ i \int d^4x \mathcal{L} \right] = \exp \left[ i \int d^4x \mathcal{L}_0 \right] \left( 1 - i \int d^4x \frac{\lambda}{4!} \phi^4 + \dots \right)$$

$\mathcal{I}_1$  and  $\mathcal{I}_2$  can now be written in terms of

$$\text{I)} \quad \begin{array}{c} p \\ \rightarrow \\ \circ \text{---} \circ \end{array} = \frac{i}{p^2 - m^2 + i\epsilon}$$

$$\text{II)} \quad \begin{array}{c} p_1 \quad p_2 \\ \swarrow \quad \nwarrow \\ \bullet \\ \nearrow \quad \searrow \\ p_3 \quad p_4 \end{array} = -i\lambda (2\pi)^4 \delta^{(4)} \left( \sum_i p_i \right),$$

$$\text{III)} \quad \text{external line} \quad \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} \leftarrow p \quad \circ \text{---} X = e^{-ipX}$$

IV) integrate over each underdetermined loop momentum:

$$\int \frac{d^4 p}{(2\pi)^4}$$

V) incorporate symmetry factors

## 5.5. Functional Derivatives and Generating Functions

more elegant way to evaluate path integrals:

use generating function

$$Z[J] = \int \mathcal{D}\phi \exp \left[ i \int d^4 x (\mathcal{L} + J(x)\phi(x)) \right]$$

source term

$$\langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle = \frac{1}{Z_0} \left( -i \frac{\delta}{\delta J(x_1)} \right) \dots \left( -i \frac{\delta}{\delta J(x_n)} \right) Z[J] \Big|_{J=0}$$

with  $Z_0 = Z[J=0]$

for the free theory:

$$Z[J] = Z_0 \exp \left[ -\frac{1}{2} \int d^4 x d^4 y J(x) D_F(x-y) J(y) \right]$$

→ EX derive Wick's theorem using this generating function.