4.4. Cartan Subalgebra

Def.: Let g be a Lie algebra, and $h \subset g$ a subalgebra. h is called an ideal when $\forall x \in g$, $y \in h$ $\quad [x, y] \in h$

subalgebra : $[h, h] \subset h$
and $\quad [g, h] \subset h$ $\Big\}$ h = ideal

(I.) An algebra is called **simple** if it does not have any proper ideals. ← strictly smaller than the whole algebra

(II.) An algebra is called **semi-simple** if it is the direct sum of simple algebras.

Example: $U \in U(N) \longrightarrow U^+ = U^{-1} \quad (\mathbb{1}^+ = \mathbb{1} = \mathbb{1}^{-1})$

$$U = \exp(i\,\delta U) \longrightarrow \delta U^+ = \delta U$$

δU → $\mathbb{1}$
δU → all other $N^2 - 1$ traceless, hermitian matrices

$[\mathbb{1}, \mathbb{1}] = 0 \cdot \mathbb{1}$ subalgebra ✓
$[\mathbb{1}, X] = 0 \cdot \mathbb{1}$ ✓ $\Big\}$ ideal

→ $\mathbb{1}$ is a proper ideal in the Lie algebra $u(N)$
→ $u(N)$ is not simple

by removing this ideal we get $SU(N)$ which is simple.
← ($U(1)$ - factor)

Cartan: An algebra is semi-simple iff (if and only if) the Killing form is non-singular.

To work with a semi-simple Lie algebra, it is convinient to use an eigenbasis $\{\tilde{T}_a\}$ with:

$$ad_X(\tilde{T}_a) = [X, \tilde{T}_a] = \xi^a \tilde{T}_a$$

eigen values for eigen vectors $\tilde{T}_a$

solve characteristic equation:

$$\det(\rho - \mathbb{1}\vec{\xi}) = 0$$

matrix representation of $ad_X(\tilde{T}_a)$

We want that solutions always $\exists$, therefore we use the field $\mathbb{C}$ (smallest algebraic complete field) instead of $\mathbb{R}$ from now on.

Finally, we need the maximal set of generators X which simultainiously solves the eigenvalue equation.

Remember quantum mechanics:

$\longrightarrow$ have to be linearly independent & commute

They $\{H_i\}$, span the abelian Cartan subalgebra with $[H_i, H_j] = 0$

The dimension of this subalgebra is called rank

$$r := \text{rank } G = \dim G_0$$

Cartan subalgebra

5. Root System

remember: last section Cartan subalgebra

generators $[H_i, H_j] = 0 \quad \forall H_i \in g_0, \ i = 1, \dots, \operatorname{rank} g$

Why? Because all element $H_i \in g_0$ can be diagonalised simultaniously

$$[H, Y] := ad_H(Y) = \alpha_Y(H)\, Y$$

eigen vector and corresponding eigen value

eigenvalues are $\boxed{\text{roots}}$ of the characteristic equation

They are assigned to each element of g_0:

$$\alpha_Y : g_0 \to \mathbb{C}$$

→ Roots are elements of the vector space g_0^* dual to g_0

Idea: decompose Lie algebra g into

$$g = g_0 \bigoplus_{\alpha \neq 0} g_\alpha \,, \quad g_\alpha = \{ Y \in g \mid [H, Y] = \alpha(H)\, Y, \forall h \in g_0 \}$$

Cartan elements — non-zero roots

This is called root space decomposition of g.

g_0 = Cartan subalgebra & g_α = root subspaces

The collection of all the roots is called root system $\Phi = \Phi(g)$. For a simple Lie algebra, it has the properties:

1) The root system spans g_0^*, $\operatorname{span}_{\mathbb{C}}(\Phi) = g_0^*$.

$\lightning$ NOT a basis because more roots than rank g

2) For any $\alpha \in \Phi$, there is a $Y_{-\alpha} \in g$ such that

Killing metric $\longrightarrow$ $K(Y_\alpha, Y_{-\alpha}) \neq 0$.

3) The only multiples of $\alpha \in \Phi$ which are roots are $\pm\alpha$.

4) The root spaces g_α are one dimensional.

Homework: Try to prove 1) - 4).

Because of them we can introduce the

5.1. Cartan-Weyl Basis

(I.) basis for g_0 $\quad H_\alpha$ such that $\forall\, H \in g_0$

$$\alpha(H) := c_\alpha\, K(H_\alpha, H)$$

normalisation constants, we fix them later

This choice gives rise to the non-degenerate pairing

$$(\alpha, \beta) := c_\alpha c_\beta\, K(H_\alpha, H_\beta) = c_\alpha\, \beta(H_\alpha) = c_\beta\, \alpha(H_\beta)$$

on g_0^*.

(II.) for each $H_\alpha \in g_0$ there is an associated root E_α with $[H, E_\alpha] = \alpha(H)\, E_\alpha$.

$\rightarrow$ we can associate to any root α an $sl(2,\mathbb{C})$ subalgebra generated by $\{E_\alpha, E_{-\alpha}, H_\alpha\}$.

Check by calculating: $\quad [E_\alpha, E_{-\alpha}]$

$$K(H, [E_\alpha, E_{-\alpha}]) = K([H, E_\alpha], E_{-\alpha})$$

ad-invariance of Killing form

$$= \alpha(H)\, K(E_\alpha, E_{-\alpha}) \neq 0 \text{ because of 2)}$$

$$= c_\alpha\, K(H, H_\alpha)$$

$K(\circ,\circ)$ is non-degenerate and eq. holds $\forall H \in \mathfrak{g}_0$

$\to$ $[E_\alpha, E_{-\alpha}] = c_\alpha K(E_\alpha, E_{-\alpha}) H_\alpha$, we also have:

$$[H_\alpha, E_{\pm\alpha}] = \pm\alpha(H_\alpha) E_{\pm\alpha} = \pm \underbrace{\frac{(\alpha,\alpha)}{c_\alpha}} E_{\pm\alpha}$$

standard normalisation for $sl(2,\mathbb{C}) \to 2$

$$c_\alpha = \tfrac{1}{2}(\alpha,\alpha) \quad \text{and therefore} \quad \alpha(H_\beta) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$$

We see that the structure of $sl(2,\mathbb{C})$ [= complexified version of $SU(2)$] occures multiple time inside a simple Lie algebra.

We already have $[H_\alpha, E_\beta]$ and $[E_\alpha, E_{-\alpha}]$, but what about $[E_\alpha, E_\beta]$?

Tool <u>root vector:</u> $\quad [H_i, E_\alpha] = \alpha_i E_\alpha$

or root for short $\quad \alpha_i := \alpha(H_i)$

$$\begin{aligned}[H_i,[E_\alpha,E_\beta]] &= -[E_\alpha,[E_\beta,H_i]] - [E_\beta,[H_i,E_\alpha]] \\ &= [E_\alpha,[H_i,E_\beta]] + [[H_i,E_\alpha],E_\beta] \\ &= (\alpha_i+\beta_i)[E_\alpha,E_\beta]\end{aligned}$$

For $\alpha+\beta \neq 0$ this implies that $[E_\alpha, E_\beta]$ is proportional to the generator $E_{\alpha+\beta}$ of the root subspace $\mathfrak{g}_{\alpha+\beta}$, provided that $\alpha+\beta \in \Phi$. To summarise:

<u>Cartan-Weyl basis</u>

$$\boxed{\begin{aligned}&[H_i, E_\alpha] = \alpha_i E_\alpha, \qquad [H_i, H_j] = 0 \\ &[E_\alpha, E_{-\alpha}] = H_\alpha = \alpha^{\vee i} H_i, \quad [E_\alpha, E_\beta] = \begin{cases} e_{\alpha,\beta} E_{\alpha+\beta}, & \alpha+\beta\in\Phi \\ 0, & 0\neq\alpha+\beta\notin\Phi\end{cases}\end{aligned}}$$

coroot for α

with the normalisation $K(E_\alpha, E_{-\alpha})\, c_\alpha = 1$

which implies $K(E_\alpha, E_\beta) = \frac{2}{(\alpha,\alpha)} \delta_{\alpha,-\beta}$.