Canonical form of h_{ij}

It is always possible to find a basis such that h_{ij} has one of the forms. For example:

$\boxed{1.}$ Symmetric sesquilinear $\quad h^{\vee} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} \quad \lambda_1, \dots \lambda_N \in \mathbb{R}$

further simplifies after rescaling to $\qquad$ real eigenvalues

$$h^{\vee} = \begin{pmatrix} +\mathbb{1}_{N_+} & 0 & 0 \\ 0 & 0 \cdot \mathbb{1}_{N_0} & 0 \\ 0 & 0 & -1 \cdot \mathbb{1}_{N_-} \end{pmatrix}$$

$N_0 > 0$ makes $h^{\vee}$ singular

$\rightsquigarrow$ non-singular metrics $h^{\vee}$ are characterised by (N_+, N_-)

signature of metric

$\boxed{2.}$ bilinear antisymmetric

Note $(h^{\wedge})^T = -h^{\wedge} \rightsquigarrow \det h^{\wedge} = \det (h^{\wedge})^T = (-1)^N \det h^{\wedge}$

or $h^{\wedge}$ can only be non-singular when N is even. //

Then: $h^{\wedge} = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \\ & & & & \ddots \end{pmatrix}$ and by normalising the basis vectors pairwise

$$h^{\wedge} = \begin{pmatrix} \varepsilon & \dots & 0 \\ 0 & & \varepsilon \end{pmatrix} = \mathbb{1}_N \otimes \varepsilon \quad \text{with} \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

3.4. Metric preserving groups

Def.: Metric preserving groups are subgroups of $GL(N, \mathbb{F})$ that preserve a certain inner product:

$e_i \longrightarrow e_i' = e_j A^j{}_i$ Einstein sum convention

$$h'_{ij} = \langle e'_i , e'_j \rangle = \langle e_k A^k{}_i , \; e_l A^l{}_j \rangle$$
$$= A^{(*)k}{}_i \, h_{kl} \, A^l{}_j \overset{!}{=} h_{ij}$$

Check: Is it a group?

closure requires: $\left(A^l{}_k B^k{}_i\right)^{(*)} = B^{(*)k}{}_i \, A^{(*)l}{}_k$

↯ True for $\mathbb{R}$ and $\mathbb{C}$ but for $\mathbb{H}$ we require complex conjugation $(q_1 q_2)^* = q_2^* q_1^*$

$q_1, q_2 \in \mathbb{H}$

metric	bilinear	sesquilinear
symmetric	$\boxed{\text{orthogonal}}$ $(h_{ij} = h_{ji})$ $O(N_+, N_-; \mathbb{R})$ $O(N_+, N_-; \mathbb{C})$ $O(N_+, N_-; \mathbb{H})$ $\Big\}$ $\dim_{\mathbb{F}} = \frac{N(N-1)}{2}$	$\boxed{\text{unitary}}$ $(h_{ij} = h^*_{ji})$ $U(N_+, N_-; \mathbb{C})$, $\dim_{\mathbb{R}} = N^2$ $U(N_+, N_-; \mathbb{H})$, $\dim_{\mathbb{R}} = (2N+1)N$
antisymmetric	$\boxed{\text{symplectic}}$ $(h_{ij} = -h_{ji})$ $Sp(2N, \mathbb{R})$ $Sp(2N, \mathbb{C})$ $\Big\}$ $\dim_{\mathbb{F}} = N(2N+1)$ $Sp(N, \mathbb{H})$ $\dim_{\mathbb{H}} = \frac{N(N+1)}{2}$	$h_{ij} = -h^*_{ji}$ for $\mathbb{R}$ $h'_{ij} = i h_{ij}$ $\quad$ $\mathbb{C}$, $\mathbb{H}$ $\Rightarrow h'_{ij} = h'^*_{ij}$

3.5. Metric & Volume preserving groups

are denoted by an S, for special metric group in front of O, U, Sp: $SO(N_+, N_-)$, $SSp(2N)$ and $SU(N_+, N_-)$

for example $SO(3)$ from 1st lecture $\quad N_+ = 3$, $N_- = 0$

4. Cartan-Weyl basis

4.1. Motivation

last section many new Lie groups:
$U(N)$, $O(N)$, $Sp(2N)$, ...

Question: • Are there more?
• How do we find representations

Problem: Dealing with Lie groups is complicated!
I.e. $SL(N)$ with $A \in M_{N\times N}$, $\det A = 1$
degree N polynomial in the matrix elements
$\longrightarrow$ find zeros of $\det A - 1 = 0$
only possible for $N \leq 4$ analytically
Solution to this problem: transistion to Lie algebra

(1.) consider adjoint action on group elements X

$$X' = A \cdot X \cdot A^{-1}$$

(2.) make this action infinitesimal around the identity element $\mathbb{1}$

$$A = \mathbb{1} + \delta A \longrightarrow A^{-1} = \mathbb{1} - \delta A$$

Check: $AA^{-1} = A^{-1}A = \mathbb{1} - (\delta A)^2$ ignored because quadratic

(3.) take the same expansion for X to find:

$$\mathbb{1} + \delta X' = (\mathbb{1} + \delta A)(\mathbb{1} + \delta X)(\mathbb{1} - \delta A)$$
$$= \mathbb{1} + [\delta A, \delta X] + \ldots \leftarrow \text{quadratic \& cubic terms}$$

$$\delta X' = [\delta A, \delta X]$$
adjoint representation of the corresponding Lie algebra

We have used: 2) associativity } from the
3) identity element } group axioms
4) inverse element }

1) closure is first encountered at quadratic order where we require the Jacobi identity

$$[A,[B,C]] = [[A,B],C] + [B,[A,C]]$$

Result: <u>Exponential map</u> $\boxed{\exp : g \to G}$

relates elements of the Lie algebra g to elements of the Lie group G

$A \in G \quad \delta A \in g$

$$\exp(\delta A) = \sum_{k=0}^{\infty} \frac{\delta A^k}{k!} = \mathbb{1} + \delta A + \frac{1}{2}(\delta A)^2 + \ldots$$

There are some information about A lost in δA. I.e. the Lie groups $SU(2)$, $SO(3)$ and $O(3)$ share the same Lie algebra but they are not equivalent.

$\longrightarrow$ <u>From now on, we consider Lie algebras</u>

4.2. Adjoint representation

Def.: Let G be a Lie group and $g, h \in G$, then the <u>adjoint action</u> is defined as

$$\boxed{Ad_g(h) = g \cdot h \cdot g^{-1}}$$

It is a group homomorphism $g \mapsto Ad_g$:

$$Ad_{g_2} \circ Ad_{g_1}(h) = Ad_{g_2}(g_1 h g_1^{-1}) = g_2 g_1 h g_1^{-1} g_2^{-1}$$

composition $\circ$ becomes the group multiplication $= Ad_{g_2 \cdot g_1}$

on the level of the Lie algebra, we have already seen

$$ad_X(Y) := [X, Y] \qquad , X, Y \in g$$

and again check:

$$[ad_{X_2}, ad_{X_1}](Y) = (ad_{X_2} \circ ad_{X_1} - ad_{X_1} \circ ad_{X_2})(Y) =$$
$$ad_{X_2}([X_1, Y]) - ad_{X_1}([X_2, Y]) = [X_2, [X_1, Y]] - [X_1, [X_2, Y]]$$
$$= [[X_2, X_1], Y] = ad_{[X_1, X_2]}(Y)$$

Jacobi identity

All information we need to fix $ad_X(Y)$ are contained in the Lie algebra's structure coefficients.

$$ad_{T_a}(T_b) = [T_a, T_b] = \sum_c f_{ab}{}^c T_c \quad \text{(see sec. 2.5)}$$

$ad_{T_a}(\cdot)$ is a representation, we find for any Lie algebra g:

$$ad_{T_a}(T_b) = \sum_c (f_a)_b{}^c T_c$$

$1, \dots, \dim G$

$(f_a)_b{}^c$ are $\dim G$ different $(\dim G) \times (\dim G)$-matrices

4.3. Killing form

There is a natural inner product on the Lie algebra called the Killing form (or metric):

$$K(X, Y) := \frac{1}{I} \operatorname{tr}(ad_X \circ ad_Y)$$

normalisation constant

trace in the adjoint matrix representation

In particular, for a matrix $\rho^a{}_b$, $\mathrm{tr}\,\rho = \sum_a \rho^a{}_a$

$$\begin{aligned} \mathrm{ad}_{T_a} \circ \mathrm{ad}_{T_b} (T_c) &= \mathrm{ad}_{T_a}([T_b, T_c]) \\ &= \sum_d f_{bc}{}^d \, \mathrm{ad}_{T_a}(T_d) \\ &= \sum_{d,e} \underbrace{f_{bc}{}^d f_{ad}{}^e}_{(\rho_{ab})_c{}^e} T_e \end{aligned}$$

$$\longrightarrow \boxed{K(T_a, T_b) = K_{ab} = \frac{1}{I} \sum_{c,d} f_{ad}{}^c f_{bc}{}^d}$$

We can check that

$K(X,Y) = K(Y,X)$ symmetric

$K([X,Y], Z) + K(Y, [X,Z]) = 0$ invariant under adjoint action

hold.