

4. Non-abelian gauge symmetries & Lie groups

so far: reminder of basics from QED

now: push these ideas to learn something new

4.1. Particles & global symmetries

Idea: find larger local symmetries (than $U(1)$)

i.e. two Dirac fermions:

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \text{ which transform as}$$

$$\Psi' = \exp\left(i \frac{\alpha^i \sigma_i}{2}\right) \Psi = V \Psi \quad \begin{array}{l} \text{2x2 Pauli matrices} \\ \text{mix spinors } \Psi_1 \text{ and } \Psi_2 \end{array}$$

⚡ $\Psi \hat{=}$ column vector with 8 components
 $\sigma_i \otimes \mathbb{1}_4$ and $(\sigma_i \Psi)^{ab} = \delta_{\beta}^{\alpha} (\sigma_i)^b_c \Psi^{\beta c}$

$$\sigma_i^{\dagger} = \sigma_i \text{ implies } V^{\dagger} = V^{-1} \leftarrow \begin{array}{l} \text{def. unitary matrix} \\ \leadsto SU(2) \end{array}$$

I. global symmetries:

$$\mathcal{L} = \bar{\Psi} (i \not{\partial}) \Psi - m \bar{\Psi} \Psi \quad \bar{\Psi} = (\Psi_1^{\dagger} \gamma_0 \quad \Psi_2^{\dagger} \gamma_0)$$

$$\bar{\Psi}' = \Psi'^{\dagger} \gamma_0 = \Psi^{\dagger} \underbrace{V^{\dagger} \gamma_0}_{\text{commute} = \mathbb{1}_2 \times \gamma_0} = \bar{\Psi} V^{\dagger}$$

$$\leadsto \mathcal{L}' = \mathcal{L} \quad \checkmark$$

II. conserved currents:

$$\delta \Psi = \alpha^i \frac{\partial}{\partial \alpha^i} \Psi \Big|_{\alpha^i=0} = i \frac{\alpha^i \sigma_i}{2} \Psi \quad (\text{infinitesimal version})$$

$$\delta S = 0 = \dots = \int d^4x \partial_{\mu} \left(\frac{\delta \mathcal{L}}{\delta (\partial_{\mu} \Psi)} \delta \Psi(\alpha^i) \right) = \int d^4x \partial_{\mu} J^{\mu}_i \alpha^i$$

⤴ see last lecture, i.e. use field equations ⤴
conserved current

$$= \int d^4x \partial_\mu \left(-\frac{\partial_i}{2} \bar{\Psi} \gamma^\mu \sigma_i \Psi \right)$$

$$\boxed{J_i^\mu = -\frac{1}{2} \bar{\Psi} \gamma^\mu \sigma_i \Psi} \quad \text{with } \partial_\mu J_i^\mu = 0$$

conserved charges $Q_i = \int d^3x J_i^0$

? physical interpretation of Q_i ?

remember canonical momentum $\pi = \frac{\delta \mathcal{L}}{\delta \dot{\Psi}} = i\Psi^\dagger$

$$Q_i = \int d^3x \frac{(i)^2}{2} \Psi^\dagger \sigma_i \Psi = \frac{i}{2} \int d^3x \pi \sigma_i \Psi$$

$$\{Q_i, Q_j\} = -\frac{1}{4} \int d^3x \int d^3y \{ \pi \sigma_i \Psi(\vec{x}), \pi \sigma_j \Psi(\vec{y}) \}$$

↖ Poisson brackets

$$= -\frac{i}{4} \int d^3x \pi [\sigma_i, \sigma_j] \Psi(\vec{x})$$

$$\boxed{[\sigma_i, \sigma_j] = 2i \varepsilon_{ij}^k \sigma_k} \quad \text{Lie algebra } su(2)$$

$$\hookrightarrow \{Q_i, Q_j\} = i \varepsilon_{ij}^k Q_k$$

↓ canonical quantisation

$$[Q_i, Q_j] = i \varepsilon_{ij}^k Q_k$$

$\hat{=}$ angular momentum with $Q_i = L_i$

Excuse "Good ol' angular momentum in QM"

we can diagonalise $L^2 = \sum_i L_i^2$ and L_3 at the same time
casimir operator ↗
↗ Cartan generator of $su(2)$

because $[L^2, L_3] = 0$

Hilbert space:

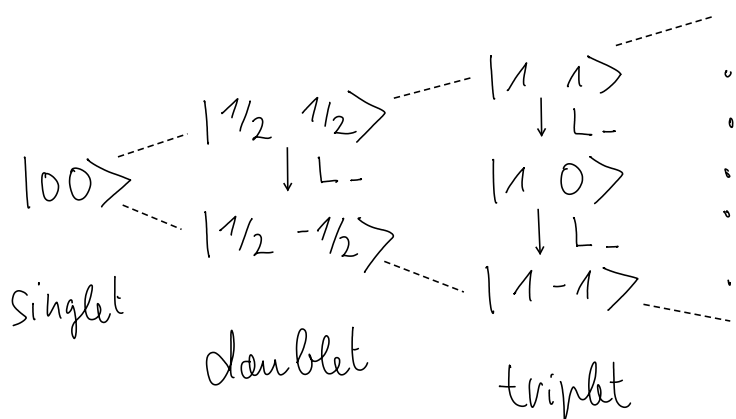
$$L^2 |l m\rangle = l(l+1) |l m\rangle$$

$$L_z |l m\rangle = m |l m\rangle$$

$$l = 0, \frac{1}{2}, 1, \dots$$

$$m = -l, -l+1, \dots, l$$

↑ spin quantum number



analog: eigenvalues of $Q_3 =$ iso spin

$$N_u \dots \text{number of up quarks} \quad Q_3 = \frac{1}{2} (N_u - N_d)$$

$N_d \dots$ number of down quarks

Doublet = fundamental representation of Lie algebra $su(2)$
 ↑ all other irreps from tensor products

i.e. $|1/2 \ 1/2\rangle = u$
 one quark $|1/2 \ -1/2\rangle = d$

two quarks, $|1 \ 1\rangle = u \bar{d}$ π^+
 spin = 0 $|1 \ 0\rangle = \frac{1}{\sqrt{2}} (u \bar{u} - d \bar{d})$ π^0
 $|1 \ -1\rangle = d \bar{u}$ π^- } mesons
 $|0 \ 0\rangle = \frac{1}{\sqrt{2}} (u \bar{u} + d \bar{d})$ ↗ EX 3.2. for details

→ Particles organise into representations of the global symmetry?

Idea of Yang & Mills gauge this symmetry
 ↑ make it local

4.2. The Yang-Mills Lagrangian

we need covariant derivative?

remember $U(1)$: $D_\mu \phi = (\partial_\mu + ie A_\mu) \phi$ ↖ connection or gauge field

now:
$$D_\mu = \partial_\mu - ig A_\mu^i \frac{\sigma_i}{2}$$

check: $\delta D_\mu \psi = \partial_\mu \delta \psi - ig \delta A_\mu^i \frac{\sigma_i}{2} \psi - ig A_\mu^i \frac{\sigma_i}{2} \delta \psi$
 $= i \alpha^i \frac{\sigma_i}{2} D_\mu \psi - \frac{1}{4} g \alpha^i A_\mu^j [\sigma_i, \sigma_j] \psi + i \partial_\mu \alpha^i \frac{\sigma_i}{2} \psi - ig \delta A_\mu^i \frac{\sigma_i}{2} \psi$
 $\stackrel{!}{=} 0$

$$\Rightarrow \delta A_\mu^i \frac{\sigma_i}{2} = \frac{1}{g} \partial_\mu \alpha^i \frac{\sigma_i}{2} + \frac{i}{4} g \alpha^i A_\mu^j [\sigma_i, \sigma_j]$$

Field strength: $[D_\mu, D_\nu] \psi = -ig F_{\mu\nu}^i \frac{\sigma_i}{2} \psi$

$$\dots F_{\mu\nu}^i \frac{\sigma_i}{2} = 2 \partial_{[\mu} A_{\nu]}^i \frac{\sigma_i}{2} - ig A_\mu^j A_\nu^k [\frac{\sigma_j}{2}, \frac{\sigma_k}{2}]$$

simplify by using: $t_i = \frac{\sigma_i}{2}$

$$[t_i, t_j] = if_{ij}^k t_k$$

results in

$$\begin{aligned} \delta A_\mu^i &= g^{-1} \partial_\mu \alpha^i - g \alpha^j A_\mu^k f_{jk}^i \\ F_{\mu\nu}^i &= 2 \partial_{[\mu} A_{\nu]}^i + g A_\mu^j A_\nu^k f_{jk}^i \end{aligned}$$

remark: - holds not just for $SU(2)$ but for any Lie group
- for $f_{ij}^k = 0$ results from last lecture

Lagrangian: should not transform $\rightarrow$ trivial representation

Killing metric: $2 \text{Tr}(t_i t_j) = K_{ij}$

compute: $2 \text{Tr} \left(\frac{0_i}{2} \frac{0_j}{2} \right) = \frac{1}{2} \text{Tr}(0_i 0_j) = \delta_{ij} = K_{ij}$
 K^{ij} inverse with $K^{ik} K_{kj} = \delta^i_j$
used to raise & lower "Lie alg. indices"

i.e. $F_{\mu\nu} = K_{ij} F^j_{\mu\nu}$

the most general Lagrangian with two derivatives is

$$\mathcal{L} = \bar{\Psi}(i \not{D} \Psi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - c \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F^i_{\alpha\beta} - m \bar{\Psi} \Psi$$

⚡ third term violates discrete P and T symmetry
parity $\rightarrow$ time reversal

$$P: X^i \rightarrow -X^i, \quad T: X^0 \rightarrow -X^0$$

$\leadsto c = 0$ (see QCD Θ -angle)

4.3. The Standard Model

gauge groups realised in nature:

$$SU(3) \times \underbrace{SU(2) \times U(1)}_{\leftarrow \text{broken by Higgs mechanism}}$$

quantum chromodynamics $\rightarrow$ electroweak interaction

$\sim$ Strong force

$\sim$ electro-magnetic & $\sim$ weak force