2.4. Linear algebra

A <u>linear algebra</u> A consists of a vector space A over a field $\mathbb{F}$ with an additional <u>vector multiplication</u> $\times$ such that:

1) $v, w \in A \rightarrow v \times w \in A$ closure
2) $(v_1 + v_2) \times w = v_1 \times w + v_2 \times w$
   $v \times (w_1 + w_2) = v \times w_1 + v \times w_2$ } bilinearity

2.5. Lie algebra

A Lie algebra g is a linear algebra with the <u>Lie bracket</u> $[\cdot, \cdot]$ as vector product, satisfying:

1) $[x, y] = -[y, x] \quad \forall x, y \in g$
2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ Jacobi identity

<u>Remarks:</u> - if the element of g can be realised as $n \times n$ matrices A, B with

$$[A, B] = A \cdot B - B \cdot A$$

then 1) and 2) hold automatically

- <u>if we do not have matrices:</u>

$$[T_a, T_b] = f_{ab}{}^c T_c$$

Structure coefficients

1) $f_{ab}{}^c = -f_{ba}{}^c$

2) $\sum_c (f_{ab}{}^c f_{cd}{}^e + f_{bd}{}^c f_{ca}{}^e + f_{da}{}^c f_{cb}{}^e) = 0$

2.6. Lie group

A Lie or <u>continuous group</u> is an n-dimensional

manifold G and a mapping $\varphi: G \times G \to G$ such that φ defines a group multiplication.
The mappings φ and $\psi: G \to G$, defined as $\psi(a) = \underset{\nwarrow \text{inverse element}}{a^{-1}}$, are both continous.

3. Classical Matrix Groups

Reminder: 1^{st} lecture example so(3)
2^{nd} lecture basic definitions

Now, many more Lie groups which are relevant in physics. They are formed (not all!) by matrices that preserve some additional structure
$\Rightarrow$ matrix groups

3.1. General Linear Group $GL(N)$

we learned: basis of a vector space is not unique
$\to \{e_i\}, \{e'_i\}, \{e''_i\} \quad i = 1, \ldots, N$
They are related by matrices $A, B, C \in Mat_{N \times N}$

$$e'_i = \sum_j e_j A^j{}_i, \quad e''_i = \sum_j e'_j B^j{}_i, \quad \text{and } e''_i = \sum_j e_j C^j{}_i$$

The group structure arises by combining them:

$$e''_i = \sum_j e_j C^j{}_i = \sum_j e'_j B^j{}_i = \sum_{j,k} e_k \underbrace{A^k{}_j B^j{}_i}_{\text{matrix multiplication}}$$

$\Rightarrow C = A \cdot B$

GL(N): $N \times N$ matrices, A, which are invertable, and therefore have $\det A \neq 0$.

Check group structure:

1) $A \cdot B$ is $N \times N$ matrix and $\det(A \cdot B) = \overset{\neq 0}{\det A} \cdot \overset{\neq 0}{\det B}$
$\neq 0$
$\rightsquigarrow$ closure ✓

2) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$, associativity ✓

3) $\mathbb{1} \cdot A = A \cdot \mathbb{1} = A \qquad (\mathbb{1})^i{}_j = \delta^i_j$, identity ✓

4) $A^{-1} \cdot A = A \cdot A^{-1} = \mathbb{1}$, inverse ✓

Note: A vector space can be defined for any field $\mathbb{F}$ and so can $GL(N)$. In particular we have:
$GL(N, \mathbb{R})$, $GL(N, \mathbb{C})$, and $GL(N, \mathbb{H})$)

Representations:

I. covariant vector: $e_i = \sum_j e_j A^j{}_i$

II. contravariant vector: $v'^i = \sum_j (A^{-1})^i{}_j v^j$

III. scalar: $\sum_i e'_i v'^i = \sum_{i,j,k} e_j \underbrace{A^j{}_i (A^{-1})^i{}_k}_{(A \cdot A^{-1})^j{}_k = \delta^j_k} v^k$

$= \sum_i e_i v^i$ (does not transform)

IV. combine vector (spaces)

a) direct sum: $e_i \oplus f_I = (e_i \; f_I) \quad e_i \in V_1, \; f_I \in V_2$

$$e'_i \oplus f'_I = \sum_{j,J} (e_j \; f_J) \begin{pmatrix} A^j{}_i & 0 \\ 0 & B^J{}_I \end{pmatrix} = \left(\sum_j e_j A^j{}_i \;\; \sum_J f_J B^J{}_I \right)$$

block diagonal matrix

b) Product: $e'_i \otimes f'_I = \sum e_j \otimes f_J \, A^j{}_i \, B^J{}_I$

tensor transformation

Now take $T'_{i_1 i_2} = \sum_{j_1, j_2} T_{j_1 j_2} A^{j_1}{}_{i_1} A^{j_2}{}_{i_2}$;

If $T_{i_1 i_2}$ is (anti-) symmetric $T'_{i_1 i_2}$ is too.

$$T_{i_1 i_2} = \pm T_{i_2 i_1}$$

$\rightarrow$ basis of $V^{\otimes 2} = V \otimes V$ splits into

$e_{i_1} \vee e_{i_2} := \frac{1}{2} (e_{i_1} \otimes e_{i_2} + e_{i_2} \otimes e_{i_1})$ and

$e_{i_1} \wedge e_{i_2} := \frac{1}{2} (e_{i_1} \otimes e_{i_2} - e_{i_2} \otimes e_{i_1})$

Note: In general, we have rank-r-tensors in $V^{\otimes r}$ with the basis $\{ e_{i_1} \otimes \ldots \otimes e_{i_r} \}$ which, among others, have totally (anti-) symmetric contributions

$$e_{i_1} \vee \ldots \vee e_{i_r} := \sum_{\sigma \in S_r} \frac{1}{r!} \sigma (e_{i_1} \otimes \ldots \otimes e_{i_r})$$

$$e_{i_1} \wedge \ldots \wedge e_{i_r} := \sum_{\sigma \in S_r} \frac{1}{r!} (-1)^\sigma \sigma (e_{i_1} \otimes \ldots \otimes e_{i_r})$$

sum over all permutations $\quad (-1)^\sigma = \begin{cases} +1 & \text{for } \sigma \text{ even} \\ -1 & \text{for } \sigma \text{ odd} \end{cases}$

3.2. Volume preserving groups SL(N)

V vector space over field $\mathbb{F}$, $\dim V = N$

The element $e_{i_1} \wedge \ldots \wedge e_{i_N}$ of $V^{\otimes N}$ is called volume element. Its standard representation

$e_1 \wedge \ldots \wedge e_N$ transforms as:

$$e'_1 \wedge \ldots \wedge e'_N = e_1 \wedge \ldots \wedge e_N \cdot (\det A)$$

remember: $\det A := \sum_{j_1, \ldots, j_N} \varepsilon_{j_1 \ldots j_N} A^{j_1}{}_1 \cdot \ldots \cdot A^{j_N}{}_N$

$\boxed{\text{SL}(N;\mathbb{F})\text{: Transformations that preserve the volume element, } A \in GL(N,\mathbb{F}) \text{ with } \det = 1 \text{, form the special linear group.}}$

↯ for quaternionic matrices A, B

$\det(A \cdot B) \neq \det A \cdot \det B$

$\rightarrow$ we only have $SL(N,\mathbb{R})$ or $SL(N,\mathbb{C})$

3.3. Metrices on Vector spaces

Def.: A <u>metric</u> or <u>inner product</u> on a vector space V maps two vectors $v, w \in V$ to a number in the associated field $\mathbb{F}$.

$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{F}$ with the properties:

1) $\langle v, w_1 a + w_2 b \rangle = \langle v, w_1 \rangle a + \langle v, w_2 \rangle b$

linearity in the 2^{nd} argument and either

2a) $\langle v_1 a + v_2 b, w \rangle = a \langle v_1, w \rangle + b \langle v_2, w \rangle$

linearity in the 1^{st} argument or

2b) $\langle v_1 a + v_2 b, w \rangle = a^* \langle v_1, w \rangle + b^* \langle v_2, w \rangle$

anti-linearity in the 1^{st} argument

$\left.\begin{array}{l} \text{bilinear} \\ \text{sesquilinear} \end{array}\right\}$ metrics

metric components are $h_{ij} := \langle e_i, e_j \rangle$

$h_{ij} = \check{h}_{ij} + \hat{h}_{ij}$ with $\check{h}_{ij} = \frac{1}{2}\left(h_{ij} + h_{ji}^{(*)}\right) = \check{h}_{ji}^{(*)}$

symmetric anti-symmetric $\qquad \hat{h}_{ij} = \frac{1}{2}\left(h_{ij} - h_{ji}^{(*)}\right) = -\hat{h}_{ji}^{(*)}$

only for sesquilinear