3. The electromagnetic (spin 1) field

3.1. Complex Klein-Gordon field

Idea: 2 real scalar fields ϕ_1, ϕ_2 $\rightsquigarrow$ 1 complex scalar ϕ

$$\phi = \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right)$$

$$\bar{\phi} = \frac{1}{\sqrt{2}}\left(\phi_1 - i\phi_2\right) \qquad \bar{\phi}_{1/2} = \phi_{1/2} \ (\text{real})$$

$$\mathcal{L}(\phi) = \mathcal{L}_{KG}(\phi_1) + \mathcal{L}_{KG}(\phi_2)$$

remember

$$\mathcal{L}_{KG}(\phi) = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2$$

$$\boxed{\mathcal{L}(\phi) = \partial_\mu\phi\,\partial^\mu\bar{\phi} - m^2\phi\bar{\phi}}$$

Field equations: treat ϕ and $\bar{\phi}$ as independent

$$\frac{\delta\mathcal{L}}{\delta\phi} = 0 \quad\Rightarrow\quad \partial_\mu\partial^\mu\bar{\phi} + m^2\bar{\phi} = 0$$

$$\frac{\delta\mathcal{L}}{\delta\bar{\phi}} = 0 \quad\Rightarrow\quad \partial_\mu\partial^\mu\phi + m^2\phi = 0$$

Note: Lagrangian $\mathcal{L}$ is invariant under

$$\phi \to e^{-i\Lambda}\phi \quad\text{and}\quad \bar{\phi} \to e^{i\Lambda}\bar{\phi}$$

Λ is a <u>real constant</u>. $\sim$ Lie group $U(1)$ $\to$ Lie algebra $u(1)$

3.2. Conserved currents and charges

infinitesimal: $\delta\phi = -i\phi\Lambda$,

$$\delta\bar{\phi} = i\bar{\phi}\Lambda$$

$$\delta S = 0 = \int d^4x \left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\mu(\delta\phi) + \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \ldots\delta\bar{\phi}\ldots\right]$$

Integration by parts

$$\overset{\triangle}{=} \int d^4x\ \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi(\Lambda) + \ldots\delta\bar{\phi}(\Lambda)\ldots\right) -$$

$$\int d^4x \left[\underbrace{\left(\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} - \frac{\delta \mathcal{L}}{\delta \phi} \right)}_{\text{field equations for } \bar{\phi} = 0 \text{ (on-shell)}} \delta\phi(\Lambda) + \dots \delta\bar{\phi}(\Lambda) \right] = 0$$

therefore $\delta S = 0 = \int d^4x \, \partial_\mu J^\mu$ with

$$\boxed{J^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta\phi(\Lambda) + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \bar{\phi})} \delta\bar{\phi}(\Lambda)}$$

here $J^\mu = i(\bar{\phi} \partial^\mu \phi - \phi \partial^\mu \bar{\phi})$

J^μ = <u>conserved current</u> with <u>$\partial_\mu J^\mu = 0$</u> under field eq.

$$0 = \int_{t_1}^{t_2} dx^0 \Big(\partial_0 \underbrace{\int d^3x \, J^0}_{Q(x^0) = Q(t)} - \underbrace{\int d^3x \, \partial_i J^i}_{= 0 \text{ because fields vanish @ } \infty} \Big)$$

$0 = Q(t_2) - Q(t_1) \quad \rightarrow \quad Q(t_2) = Q(t_1)$ or $Q(t) = \text{const}$

$\boxed{Q(t) = \int d^3x \, J^0}$ conserved charge

Noether's theorem: Symmetry $\Longleftrightarrow$ conserved charges

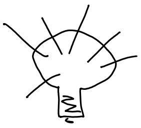
<u>3.3. Gauge symmetries</u>

- parameter Λ does not depend on X^μ!

$\rightarrow$ global symmetry (everywhere the same)

<u>Question:</u> Can we make it local, i.e. $\Lambda(X^\mu)$?

But then $\delta \mathcal{L} = \dots = (\partial_\mu \Lambda) J^\mu \neq 0$

 <u>gauge</u> $\mathcal{L}$ (compensate with additional terms)

$$\mathcal{L}_1 = -e J^\mu A_\mu$$ ← new field called gauge field with

$$\boxed{\delta A_\mu = \frac{1}{e}\partial_\mu \Lambda}$$

$$\delta \mathcal{L}_1 = -e\,\delta J^\mu A_\mu - J^\mu \partial_\mu \Lambda$$ ← term that we want

↑ but we also get this one :-c

$$\delta J^\mu = 2\bar{\phi}\phi\partial^\mu \Lambda \quad \rightarrow \quad \delta(\mathcal{L}+\mathcal{L}_1) = -2eA_\mu\partial^\mu\Lambda\phi\bar{\phi}$$

compensate again!

$$\mathcal{L}_2 = e^2 A_\mu A^\mu \phi\bar{\phi} \quad \rightarrow \quad \delta\mathcal{L}_2 = 2eA_\mu\partial^\mu\Lambda\phi\bar{\phi} \quad :-)$$

$$\begin{aligned}\mathcal{L}\text{gauged} &= \mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2 \\ &= (\partial_\mu\phi + ieA_\mu\phi)(\partial^\mu\bar{\phi} - ieA^\mu\bar{\phi}) - m^2\phi\bar{\phi}\end{aligned}$$

↯ Symmetry of this Lagrangian is hard to see. Can we do better?

Covariant derivatives:

$$\boxed{D_\mu\phi = (\partial_\mu + ieA_\mu)\phi}$$ with

$$\begin{aligned}\delta(D_\mu\phi) &= \delta(\partial_\mu\phi) + ie\,\delta A_\mu\phi + ieA_\mu\delta\phi \\ &= -i\partial_\mu(\Lambda\phi) + i\cancel{e}\frac{1}{\cancel{e}}\partial_\mu\Lambda\phi + eA_\mu\Lambda\phi \\ &= -i\Lambda(\partial_\mu\phi + ieA_\mu\phi) = -i\Lambda(D_\mu\phi)\end{aligned}$$

$D_\mu\phi$ transforms like ϕ, namely covariantly

$$\boxed{\mathcal{L}\text{gauged} = D_\mu\phi\,\overline{D^\mu\phi} - m^2\phi\phi}$$

Question: Can we generate more covariant quantities?

Yes! i.e.

$$[D_\mu, D_\nu]\phi = (\partial_\mu + ie A_\mu)(\partial_\nu + ie A_\nu)\phi - (\mu \leftrightarrow \nu)$$

$$= \cancel{\partial_\mu \partial_\nu \phi} + ie\partial_\mu(A_\nu \phi) + \cancel{ie A_\mu \partial_\nu \phi} - ie^2 A_\mu A_\nu \phi - (\mu \leftrightarrow \nu)$$

$$= ie \underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{F_{\mu\nu}} \phi$$

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu} = \text{electromagnetic field tensor}$$

$$\delta F_{\mu\nu} = 2 \frac{1}{e} \partial_{[\mu} \partial_{\nu]} \Lambda = 0 \qquad X_{[\mu\nu]} = 1/2 (X_{\mu\nu} - X_{\nu\mu})$$

$$\boxed{\mathcal{L}_{tot} = -1/4 F_{\mu\nu} F^{\mu\nu} + D_\mu \phi \overline{D^\mu \phi} - m^2 \phi \bar{\phi}}$$

Kinetic term with two derivatives

3.4. The QED Lagrangian

$$\psi(x) \to e^{i\alpha(x)} \psi(x) \qquad A_\mu \to A_\mu - \frac{1}{e} \partial_\mu \alpha(x)$$

symmetry of Dirac Lagrangian for $\alpha(x) = \text{const.}$

$$D_\mu = \partial_\mu + ie A_\mu$$

$$\boxed{\mathcal{L}_{QED} = -1/4 F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi}$$

3.5. Quantisation of A_μ

$\mathcal{L} = -1/4 F_{\mu\nu} F^{\mu\nu}$ will not change under

$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$ (gauge transformation)

This redundancy is a problem for quantisation!

Remove it by gauge fixing

1. Lorentz gauge $\partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \Lambda = 0$

$$\rightarrow \quad \boxed{\partial_\mu \partial^\mu \Lambda = -\partial_\mu A^\mu} \quad (1)$$

still not unique

(2.) Coulumb gauge $\quad A'^0 = A^0 + \underbrace{\partial^0 \Lambda}_{\frac{\partial}{\partial t}\Lambda} = 0$

$\rightarrow \boxed{\frac{\partial}{\partial t}\Lambda = -A^0}$ together with (1) we have

$$A_0 = A^0 = 0 \quad \text{and} \quad \partial_i A^i = 0$$

reduces from 4 degrees of freedom in A_μ to 2

Quantisation:

1.) conjugate momentum for A_μ:

$$\pi^0 = \frac{\delta \mathcal{L}}{\delta \dot{A}_0} = 0 \qquad \pi^i = \frac{\delta \mathcal{L}}{\delta \dot{A}_i} = -\dot{A}^i + \partial^i A^0 = F^{0i} = E^i$$

Electric field strength

2.) canonical commutator:

$$[A^i(\vec{x}), E^j(\vec{y})] = i\int \frac{d^3k}{(2\pi)^3}\left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2}\right) e^{i\vec{k}(\vec{x}-\vec{y})}$$

becomes the $\delta^{ij}\delta(\vec{x}-\vec{y})$ we know new! why?

because $\partial_i A^i = 0$

and therefore $\quad [\partial_i A^i(\vec{x}), E^j(\vec{y})] = \frac{\partial}{\partial x^i}[A^i(\vec{x}), E^j(\vec{y})]$

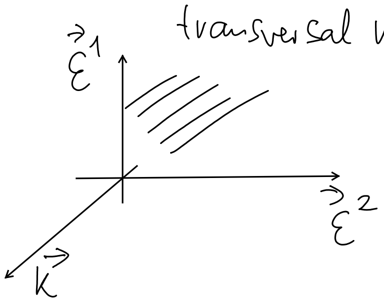
EX 2.2 requires the additional term $= 0$

3.) Mode expansion:

$$\vec{A}(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{\lambda=1}^{2} \vec{\varepsilon}^{\,\lambda}(k)\left[a_k^\lambda e^{-ikx} + (a_k^\lambda)^\dagger e^{ikx}\right]$$

polarisation vectors

$$\vec{\nabla}\vec{A} = \partial_i A^i = 0 \quad \rightsquigarrow \quad \boxed{\vec{k}\cdot\vec{\varepsilon}^{\,\lambda} = 0}$$



$\left(\vec{\varepsilon}^{\lambda} \cdot \vec{\varepsilon}^{\varrho} = \delta^{\lambda\varrho} \right)$

with $[a^{\lambda}_{\vec{k}}, (a^{\varrho}_{\vec{k}'})^{+}] = 2k_0 (2\pi)^3 \delta^{\lambda\varrho} \delta(\vec{k}-\vec{k}')$

again creation and annihilation operators of HO

4.) Hamiltonian

$$H = \frac{1}{2} \int d^3x \left(\dot{\vec{A}}^2 + (\vec{\nabla} \times \vec{A})^2 \right)$$

$$\vdots$$

$$= \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3 k_0} \frac{k_0}{2} \left((\vec{a}^{\lambda}_{\vec{k}})^{+} \vec{a}^{\lambda}_{\vec{k}} + \text{vacuum energy} \right)$$