Lie Algebras and Lie Groups

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- lecture notes and Mathematica notebooks on https://www.fhassler.de/teaching/ws_23/lie
- lectures on Tuesday 12:15-14:00, room 447
- Written exam at the end of semester
  - practice exam before

  ⇒ attendance and active participation in the lectures are crucial to pass

- Mathematica is used for some computations

  a) · student license 800 zł
     · for one semester 250 zł :-(

  b) check for university licenses
     i.e. computer lab of Institute of Mathematics

  c) free Wolfram Engine for Developers
     + Jupyter Notebook :-)
     ⇒ install instructions on the website

- Course is based on a course by Prof. Stefan Groot Nibbelink held in 2014 @ LMU Munich

↗ Literature at the website

1. Introduction

1.1. Why do we care?

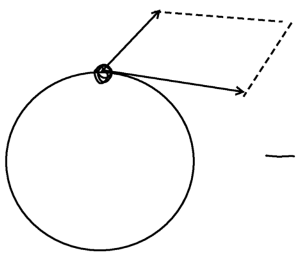
- (Lie) groups characterise symmetries in physical systems

classical mechanics: - help to solve equations of motion

quantum mechanics: - quantisation of angular momentum → Lie group $SU(2)$
- quantum numbers l and m

quantum field theory
- Lorentz and Poincaré group → spin of particles
- (gauge) groups characterise particle content of the standard model

- Lie Groups parameterised by continous set of variables = Coordinates on manifold



tangent space = Lie algebra
- encodes already most aspects of the Lie group
- much easier to deal with (just linear algebra)

- complete classification of semi simple Lie algebras (later in the course)

- Lie groups / algebras are not directly visible in physics
→ we see only representations

i.e. Lorentz group, $SO(3,1)$, acts

- on Scalars (trivially) ϕ ← Higgs boson
- on Vectors V^μ ← boson, like photon
- on Spinor Ψ

representations decompose into fundamental building blocks = irreducible repr. = irreps

1.2. Rotations in 3 dim. as example

defined by: $R^T \cdot R = \mathbb{1}_3$, $\det R = 1$ ⇝ $SO(3)$
$R \in M_{3\times 3}(\mathbb{R})$, real 3x3 matrices

like: $R_1(\alpha_1)\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,

$R_2(\alpha_2) = \begin{pmatrix} \cos\alpha_2 & 0 & \sin\alpha_2 \\ 0 & 1 & 0 \\ -\sin\alpha_2 & 0 & \cos\alpha_2 \end{pmatrix}$, and $R_3(\alpha_3) = \begin{pmatrix} \cos\alpha_3 & \sin\alpha_3 & 0 \\ -\sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

a general rotation can be written as:

$R = R_1(\alpha_1)\, R_2(\alpha_2)\, R_3(\alpha_3)$

check: 3x3 matrix R has $3\cdot 3 = 9$ real parameters

$(R^T \cdot R)^T = (\mathbb{1}_3)^T = R^T R = \mathbb{1}_3$

↳ $\frac{1}{2}\cdot 3(3+1) = 6$ constraints

$\det R = 1$ $\quad \det(R^T \cdot R) = \det(\mathbb{1}_3) = 1 = \det R^T \det R$

⇝ $\det R = \pm 1$ discrete choice $= (\det R)^2$

$\underset{\text{param.}}{9} - \overset{\text{constr.}}{6} = 3$ ⟵ dimension of Lie group $SO(3)$

$\det R = 1$ $\qquad\qquad \det R = \pm 1$

$SO(3) \subset O(3)$

special orthogonal group

1.2.1. $so(3)$ Lie algebra

tangent space of $SO(3)$ at the identity element

$$\mathbb{1}_3 = R(0,0,0) = R(\vec{0})$$

spaned by
$$\boxed{E_i = \frac{\partial}{\partial \alpha_i} R(\vec{\alpha})\Big|_{\vec{\alpha}=0}}$$

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad E_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

They span the <u>vector space</u> of antisymmetric, real 3×3 matrices

<u>check:</u> $E^T = -E$ $\quad \frac{1}{2}\,3(3+1) = 6$ constraints

$9 - 6 = 3$ dimensional ✓

Lie algebra = Vector space V + product $V \times V \to V$

naive product would be matrix product

$$E_1 \cdot E_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ ↯ not anti symmetric

⇝ correct product is the commutator

$$[E_i, E_j] = E_i \cdot E_j - E_j \cdot E_i = \sum_{k=1}^{3} \varepsilon_{ijk} E_k$$

Levi-Civita symbol,
totally anti-symmetric with $\varepsilon_{123} = 1$

$$[E_1, E_2] = \overset{=1}{\varepsilon_{123}}\, E_3$$
$$[E_3, E_1] = \varepsilon_{312}\, E_2 = \overset{=1}{\varepsilon_{123}}\, E_2$$
$$[E_2, E_3] = \varepsilon_{231}\, E_1 = \overset{=1}{\varepsilon_{123}}\, E_1$$

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