

9. Gravity and String Theory

Today more about my research. ☺

9.1 General Relativity

Idea: Construct a theory which is invariant under coordinate changes.

$$x^\mu \rightarrow x^\mu + \xi^\mu \quad \phi(x^\mu) \rightarrow \phi(x^\mu) + \xi^\nu \partial_\nu \phi(x^\mu)$$

infinitesimal & coordinate dependent → local symmetry

vector $v^\mu \rightarrow v^\mu + L_\xi v^\mu$

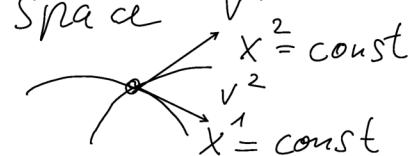
$$L_\xi v^\mu = \xi^\nu \partial_\nu v^\mu - \cancel{v^\nu \partial_\nu \xi^\mu}$$

\uparrow transformation of the tangent space

Lie derivative

scalar $\phi \rightarrow \phi + L_\xi \phi$

$$L_\xi \phi = \xi^\mu \partial_\mu \phi$$



Leibniz rule: $L_\xi (v^\mu w_\mu) = (L_\xi v^\mu) w_\mu + v^\mu (L_\xi w_\mu)$

scalar vector one-form

one-form $w_\mu \rightarrow w_\mu + L_\xi w_\mu$

$$L_\xi w_\mu = \xi^\nu \partial_\nu w_\mu + w_\nu \partial_\mu \xi^\nu$$

While $S_\xi \{\phi, v^\mu, w_\mu\} = L_\xi \{\phi, v^\mu, w_\mu\}$

$$S_\xi (\partial_\mu v^\nu) = \partial_\mu (S_\xi v^\nu) \neq L_\xi (\partial_\mu v^\nu)$$

We know this problem from Gauge theories.

→ Solution: covariant derivative

$$\boxed{\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma_{\mu\lambda}^\nu v^\lambda}$$

connection



calculate field strength with

$$[\nabla_\mu, \nabla_\nu] v^\lambda := R^\lambda_{\mu\nu\rho} v^\rho - T_{\mu\nu}^\lambda \nabla_\lambda v^\rho$$

two contributions:

$$\text{torsion } T_{\mu\nu}{}^\lambda = 2 \Gamma_{[\mu\nu]}^\lambda \quad \text{and}$$

$$\underline{\text{curvature}} \quad R^S{}_{\mu\nu} = 2(\partial_{[\mu} \Gamma_{\nu]}^S + \Gamma_{[\mu}^\lambda \Gamma_{|\nu]}^\lambda)$$

Similar to the field strength $F_{\mu\nu}^i$ in Yang-Mills
↗ Section 4.2.

Question: How do we compute $\Gamma_{\mu\nu}^S$?

Answer: I) Set torsion $T_{\mu\nu}{}^\lambda = 0$

II) require $\nabla_S g_{\mu\nu} = 0$ (metric compatible)
 $= \partial_S g_{\mu\nu} - \Gamma_{S\mu}^\lambda g_{\lambda\nu} - \Gamma_{S\nu}^\lambda g_{\mu\lambda}$

$$\hookrightarrow \boxed{\Gamma_{\mu\nu}^S = \frac{1}{2} g^{S\lambda} (\partial_\mu g_{\nu S} + \partial_\nu g_{S\mu} - \partial_S g_{\mu\nu})}$$

Levi-Civita connection

Two more important quantities:

Ricci tensor: $R_{\mu\nu} = R^S{}_{\mu S\nu}$ and

Ricci scalar: $R = R_{\mu\nu} g^{\mu\nu} = R_{\mu}{}^{\mu}$

used to construct invariant

$$S_{EH} = \frac{1}{2\lambda} \int d^4x \sqrt{-\det(g_{\mu\nu})} R$$

Einstein-Hilbert action

9.2. Quantisation

Superficial degree of divergence $D = (d-2)L + 2(1-N)$
 \nwarrow number of loops \nearrow external legs

↗ non-renormalisable for $d > 2$

↪ look at $d=2$

$$S = \frac{1}{4\pi d!} \int d^2\theta \sqrt{-g} \left(\det g_{\mu\nu} (R + \partial_\mu X^i \partial^\mu X^j G_{ij}) \right)$$

Einstein-Hilbert action in 2d does not depend on metric!

~ topological

To have propagating, local degrees of freedom add scalar fields X^i $i=1, \dots, n$ with couplings $G_{ij}(x)$

After gauge fixing of $g_{\mu\nu}$:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma G_{ij}(x) \partial_\mu X^i \partial^\mu X^j \quad \text{with } g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

non-linear ϕ -model

9.3. One-loop β -functions

We need perturbative expansion:

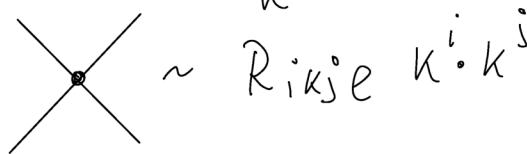
 Pick any point \bar{x}^i and expand $G_{ij}(x)$ around it.

$$G_{ij}(x) = \delta_{ij} - \frac{\alpha'}{3} R_{ikje}(\bar{x}) y^k y^e + O(y)$$
$$x^i = \bar{x}^i + \sqrt{\alpha'} y^i$$

$\stackrel{!}{=} \text{adopt Riemann normal coordinates}$

$$\rightsquigarrow S = \frac{1}{4\pi} \int d^2\sigma \left[\partial_\mu y^i \partial^\mu y^j \delta_{ij} - \frac{\alpha'}{3} R_{ikje} y^k y^e \partial_\mu y^i \partial^\mu y^j \right]$$

Feynman rules:  $\sim \frac{\delta^{ij}}{K^2}$



one-loop correction to propagator:

$$\text{---} \sim R_{ikje} \int \frac{d^d K}{(2\pi)^d} \frac{\delta^{ij}}{K^2} \sim \frac{R_{ij}}{\epsilon} + \text{finite}$$

dim. reg.

↓ apply our tools for one-loop β -function

$$\beta_{ij}(G) = d^i R_{ij}$$

conformal fix point at $R_{ij} = 0 \Leftrightarrow \delta S_{EH} = 0$

higher loop corrections to β -function

give higher derivative corrections to EH action.

Interpretation of σ -model

