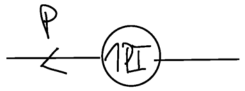
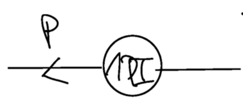
8.2. The Callan-Symanzik Equation

Problem: Cutoff Λ usually break symmetries of the theory. We know better regularisation methods, like dim. reg. $\longrightarrow$ use them! How?

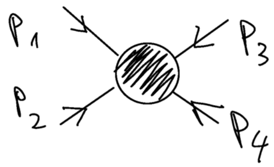
Remember renormalisation conditions

different from $s = 4m^2$ and $t = u = 0$ for 7.2. but better for calc. here

 $= 0$

$\frac{d}{dp^2}\Big($  $\Big) = 0$

} at $p^2 = -M^2$

 $= -i\lambda$ at $(p_1+p_2)^2 = (p_1+p_3)^2 = (p_1+p_4)^2 = -M^2 = s = t = u$

M is called renormalisation scale $\nearrow$ EX 11

⚡ We currently just discuss massless theories, masses later.

Questions: How does a change of M affects Green's functions?

$$G^{(n)}(x_1,\dots,x_n) = \langle 0|T\,\phi(x_1)\dots\phi(x_n)|0\rangle_{\text{connected}}$$

$M \to M + \delta M \qquad \phi \to (1+\delta\eta)\,\phi$

$\lambda \to \lambda + \delta\lambda$

$$G^{(n)}(x_1,\dots,x_n) \to (1+n\,\delta\eta)\,G^{(n)}$$

Function of M and λ

$$dG^{(n)}(x_1,\dots,x_n) = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda = n\,\delta\eta\, G^{(n)}$$

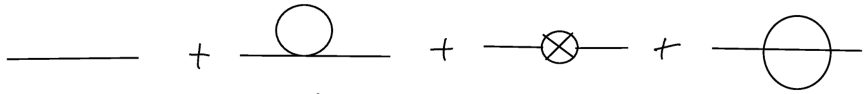
$$\beta = \frac{M}{\delta M}\delta\lambda \qquad \gamma = -\frac{M}{\delta M}\delta\eta$$

$$\Leftrightarrow \boxed{\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right]G^{(n)}(x_1,\dots,x_n;M,\lambda) = 0}$$

Callan-Symanzik equation

Example: ϕ^4-theory

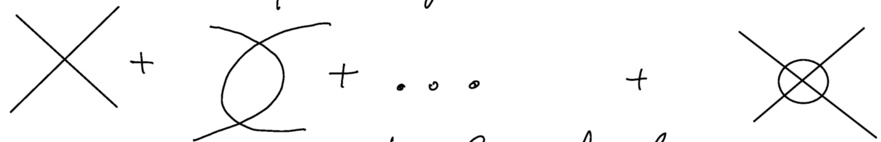
$$G^{(2)}(p) =$$



+ ...

remember: wave function renormalisation in ϕ^4 theory just @ two loops

$$G^{(4)} =$$



$+ \mathcal{O}(\lambda^3)$

s-, t- & u-chanel

$$\ldots = -i\lambda + (-i\lambda)^2 \left[i V(s) + i V(t) + i V(u) \right] - i \delta_\lambda$$

$$\delta_\lambda = (-i\lambda)^2 \, 3 V(-M^2) = \frac{3\lambda^2}{2(4\pi)^{d/2}} \int_0^1 dx \, \frac{\Gamma(2 - d/2)}{(x(1-x)M^2)^{2-d/2}}$$

renormalisation condition $s = t = u = -M^2$

$$\Rightarrow \delta_\lambda = \frac{3\lambda^2}{2(4\pi)^2} \left[\frac{1}{2 - d/2} - \log M^2 + \text{finite} \right] \quad \text{for } m = 0$$

We can now evaluate:

1) $M \frac{\partial}{\partial M} G^{(4)} = M \frac{\partial}{\partial M} (-i \delta_\lambda) = \frac{3 i \lambda^2}{(4\pi)^2}$ and

2) $\frac{\partial}{\partial \lambda} G^{(4)} = -i$ therefore

$$\left[M \frac{\partial}{\partial M} + \beta(\lambda) \frac{\partial}{\partial \lambda} + 4 \gamma(\lambda) \right] G^{(4)} = 0 \quad \text{implies}$$

$$\beta(\lambda) - 4 i \gamma(\lambda) = \frac{3\lambda^2}{(4\pi)^2} + \mathcal{O}(\lambda^3)$$

$$\left[M \frac{\partial}{\partial M} + \beta(\lambda) \frac{\partial}{\partial \lambda} + 2 \gamma(\lambda) \right] G^{(2)} = 0$$

first non-trivial contribution at two loops

$$\longrightarrow \gamma(\lambda) = 0 + \mathcal{O}(\lambda) \quad \text{and} \quad \beta(\lambda) = \frac{3\lambda^2}{(4\pi)^2} + \mathcal{O}(\lambda^3)$$

For a generic massless scalar theory:

$$G^{(2)}(p) = \text{———} + \left(\text{one loop diagrams}\right) + \text{——}\otimes\text{——} + \dots$$

$$= \frac{i}{p^2} + \frac{i}{p^2}\left(A \log\frac{\Lambda^2}{-p^2} + \text{finite}\right) + \frac{i}{p^2}\left(i p^2 \delta_Z\right)\frac{i}{p^2} + \dots$$

at leading order $O(\lambda)$ we can neglegt $\beta(\lambda)$ and find

$$\gamma = \frac{1}{2} M \frac{\partial}{\partial M}\delta_Z = \boxed{-A = \gamma}$$

$$G^{(2)} = \begin{pmatrix}\text{tree-level}\\ \text{diagrams}\end{pmatrix} + \begin{pmatrix}\text{1 PI loop}\\ \text{diagrams}\end{pmatrix} + \begin{pmatrix}\text{Vertex}\\ \text{counterterm}\end{pmatrix} + \begin{pmatrix}\text{external leg}\\ \text{corrections}\end{pmatrix}$$

$$= \left(\prod_i \frac{i}{p_i^2}\right)\Big[-ig - iB\log\frac{\Lambda^2}{-p^2} - i\delta_g + (-ig)\sum_i\left(A_i \log\frac{\Lambda^2}{-p_i^2} - \delta_{Z_i}\right)$$

$$+ \text{finit terms}\Big]$$

invariant build from p_i's all $= -M^2$ by renormalisation condition

to lowest order we then find:

$$\beta(g) = M\frac{\partial}{\partial M}\left(-\delta_g + \frac{1}{2} g \sum_i \delta_{Z_i}\right) = \boxed{-2B - g\sum_i A_i = \beta(g)}$$

Interpretation of $\gamma(\lambda)$ and $\beta(\lambda)$

remember: $\phi = Z(M)^{-1/2}\phi_0 \longleftarrow$ bare

renormalised

$$\delta\eta = \frac{Z(M+\delta M)^{-1/2}}{Z(M)^{-1/2}} - 1 \rightsquigarrow \gamma(\lambda) = \frac{1}{2}\frac{M}{Z}\frac{\partial}{\partial M} Z$$

Same argument for $\beta \rightsquigarrow \beta(\lambda) = M\frac{\partial}{\partial M}\lambda$

rates of change of normalisation (γ) and couplings (β) when energy scale M changes

8.3 Evolution of the Coupling Constants

Orginal motivation: Integrate out one mass shell after an other. $\rightsquigarrow$ renormalisation group flow

Callan - Symanzik equation = differential eq.
Solve (integrate) it!

$$G^{(n)}(p,\lambda) = \hat{G}^{(n)}\left(\bar\lambda(p;\lambda)\right) \cdot \exp\left(- \int_{p'=M}^{p'=p} d\log\left(\frac{p'}{M}\right) \cdot n\left[1-\gamma\left(\bar\lambda(p';\lambda)\right)\right]\right)$$

initial condition (measured at $p^2 = -M$)

$\bar\lambda$ = running couplings which solve:

$$\frac{d}{d\log\left(\frac{p}{M}\right)} \bar\lambda(p;\lambda) = \beta(\bar\lambda) \quad \text{and} \quad \bar\lambda(M;\lambda) = \lambda$$

$\beta(\bar\lambda) = 0$ is a fixed point of the flow