remember: last time superficial degree of divergence D in four dimensions

in d dimensions we find

$$D = dL - P_e - 2P_\gamma = \dots \nearrow \boxed{EX\ 9.b}$$

$$= d + \left(\frac{d-4}{2}\right) V - \left(\frac{d-2}{2}\right) N_\gamma - \left(\frac{d-1}{2}\right) N_e$$

three options:

a) $d < 4$ super-renormalisable: Only a finite number of Feynman diagrams superficially diverges.

b) $d = 4$ renormalisable: Only a finite number of amplitudes sup. diverges; but divergences occure at all orders in perturbation theory.

c) $d > 4$ non-renormalisable: All amplitudes are divergent at a sufficiently high order.

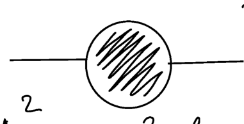
other example $\mathcal{L} = 1/2 (\partial_\mu \phi)^2 - 1/2 m^2 \phi^2 - \frac{\lambda}{n!} \phi^n$

$$D = d - \underbrace{\left[d - \left(\frac{d-2}{2}\right) n\right]}_{\text{mass dim. of coupling}} V - \underbrace{\left(\frac{d-2}{2}\right)}_{\text{mass dim. of } \phi} N$$

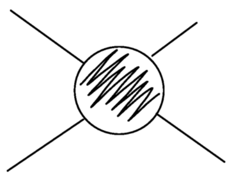
$d = 4$, $n = 4$ renormalisable

7.2. Renormalised $\varphi 4$-theory

$D = 4 - N$ $\quad \mathbb{Z}_2$-sym $\phi \to -\phi \rightsquigarrow$ diagrams with odd N vanish

a)  $D = 4$ b)  $D = 2$

Vacuum shift $\sim \Lambda^2 + p^2 \log \Lambda +$ finite

c)  $D = 0$

$\sim \log \Lambda +$ finite

3 ∞ constants adsorb them into bare mass, coupling and field strength

1. rescale the field strength

$$\phi = Z^{1/2}\phi_r \quad \nearrow \text{ see EX 8.i)}$$

$$\mathcal{L} = \tfrac{1}{2} Z(\partial_\mu \phi_r)^2 - \frac{1}{2} m_0^2 Z \phi_r^2 - \frac{\lambda_0}{4!} Z^2 \phi_r^4$$

2. eliminate m_0, λ_0 from $\mathcal{L}$ by

$$\delta_Z = Z - 1, \quad \delta_m = m_0^2 Z - m^2, \quad \delta_\lambda = \lambda_0 Z^2 - \lambda$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_r)^2 - \tfrac{1}{2} m^2 \phi_r^2 - \frac{\lambda}{4!}\phi_r^4$$

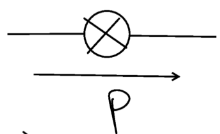
$$\boxed{+ \frac{1}{2}\delta_Z (\partial_\mu \phi_r)^2 - \frac{1}{2}\delta_m \phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4}$$

counterterms; adsorb ∞, but unobservable, shift between bare parameters and physical ones

Additional Feynman rules:

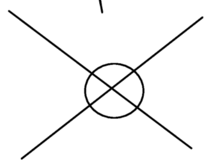
a) Propagator: $D_F(p) = \dfrac{i}{(1+\delta_Z)p^2 - (1+\delta_m)m^2}$

$$D_F(p) \approx \frac{i}{p^2 - m^2}\left[1 + i(\delta_Z p^2 - \delta_m)\frac{i}{p^2 - m^2}\right]$$



$$= i(p^2\delta_Z - \delta_m)$$

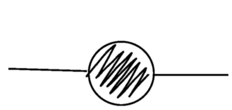
b)



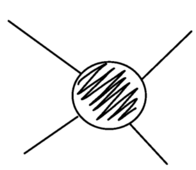
$$= -i\delta_\lambda$$

both with amputated legs

Prescription: Include these counter terms and adjust δ_Z, δ_m and δ_λ such that



$$= \frac{i}{p^2 - m^2} + \text{terms reg at } p^2 = m^2 \quad \text{and}$$

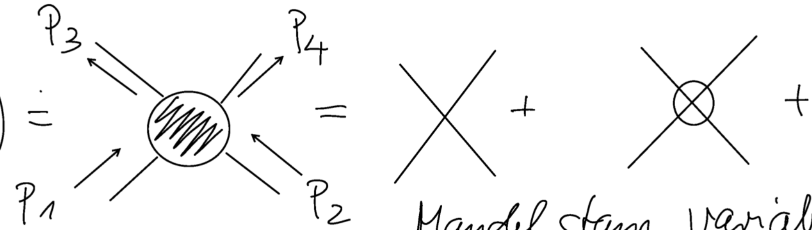


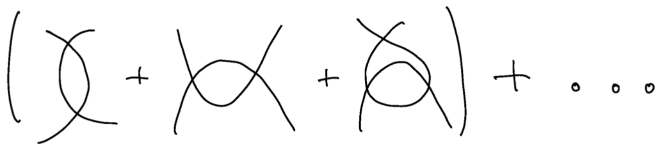
$$= -i\lambda$$

at $s = 4m^2$, $t = u = 0$

hold. The conditions are called renormalisation conditions.

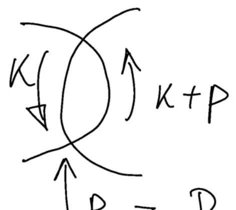
At one loop:

$i\mathcal{M}(p_1\, p_2 \to p_3\, p_4) =$ 

 $+ \ldots$

Mandelstam variables

$$s = (p_1+p_2)^2 = (p_3+p_4)^2$$
$$t = (p_1-p_3)^2 = (p_4-p_2)^2$$
$$u = (p_1-p_4)^2 = (p_3-p_2)^2$$



$$= \frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2}\,\frac{i}{(k+p)^2-m^2}$$

$$p = p_1+p_2 \qquad =: (-i\lambda)^2 \cdot iV(p^2)$$

$$i\mathcal{M} = -i\lambda + (-i\lambda)^2\left[iV(s)+iV(t)+iV(u)\right] \underbrace{-i\delta_\lambda}_{\text{counter term}}$$

$$i\mathcal{M}\Big|_{s=4m^2,\,t=u=0} \overset{!}{=} -i\lambda \longrightarrow \delta_\lambda = -\lambda^2\left[V(4m^2)+2V(0)\right]$$

after dimensional regularisation with $\varepsilon = 4-d$

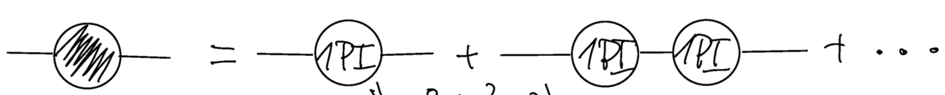
$$V(p^2) = -\frac{1}{32\pi^2}\int_0^1 dx\left(\frac{2}{\varepsilon}-\gamma+\log(4\pi)-\log\left[m^2-x(1-x)p^2\right]\right)$$

$$\delta_\lambda = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\varepsilon}-3\gamma+3\log(4\pi)-\log\left[m^2-x(1-x)4m^2\right]-2\log m^2\right)$$

$$i\mathcal{M} = -i\lambda - \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\log\left(\frac{m^2-x(1-x)s}{m^2-x(1-x)4m^2}\right)+\log\left(\frac{m^2-x(1-x)t}{m^2}\right)+\log\left(\frac{m^2-x(1-x)u}{m^2}\right)\right]$$

divergence free!

remember: for the two point function we found:

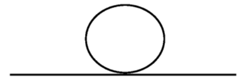
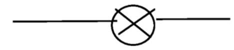


$-iM^2(p^2)$, self-energy

geometric series

$$= \frac{i}{p^2-m^2-M^2(p^2)} \overset{!}{=} \text{have pole with residue 1 at } p^2=m^2$$

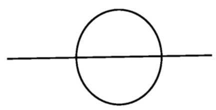
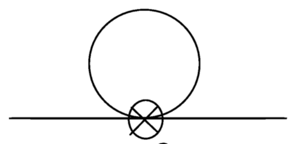
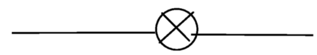
$$\longrightarrow M^2(p^2)\big|_{p^2=m^2} = 0 \text{ and } \frac{d}{dp^2}M^2(p^2)\big|_{p^2=m^2} = 0$$

at one-loop : $-iM^2(p^2) =$  + 

$$= \underbrace{-\frac{i\lambda}{2}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(1-d/2)}{(m^2)^{1-d/2}}}_{= -i\lambda/2 \cdot \int \frac{d^dk}{(2\pi)^d}\cdot\frac{i}{k^2-m^2}} + i(p^2\delta_Z - \delta_m)$$

$\longrightarrow$ $\delta_Z = 0$ and $\delta_m = -\dfrac{\lambda}{2(4\pi)^{d/2}}\dfrac{\Gamma(1-d/2)}{(m^2)^{1-d/2}}$

first contribution at two-loops from

 +  + 

$\sim \lambda^2$ $\quad\quad$ $\sim \lambda^2$