# Feynman diagrams and path integral expansion

Consider $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{4!}\phi^4$.

We want to compute:

$$\langle \phi_1 \phi_2 \rangle = \langle 0 | T\, \phi(x_1)\phi(x_2) | 0 \rangle$$
$$= \frac{\int \mathcal{D}\phi\, e^{iS}\, \phi_1 \phi_2}{\int \mathcal{D}\phi\, e^{iS}} = \frac{I_2}{I_0}$$

$$I_0 = \int \mathcal{D}\phi\, e^{iS_0}\left(1 - \frac{i\lambda}{4!}\int d^4y\, \phi^4(y) + \mathcal{O}(\lambda^2)\right)$$
$$= Z_0[0]\left(1 \underbrace{- \frac{i\lambda}{4!}\int d^4y \langle \phi^4(y)\rangle_0}_{-C} + \mathcal{O}(\lambda^2)\right)$$

with $Z_0[0] = \int \mathcal{D}\phi\, e^{iS_0}$ and $\langle \ldots \rangle_0 = \frac{\int \mathcal{D}\phi\, e^{iS_0} \ldots}{Z_0[0]}$

$$I_2 = \underset{\text{same as for } I_0}{\ldots} = Z_0[0]\left(\underbrace{\langle \phi_1 \phi_2 \rangle_0}_{A} \underbrace{- \frac{i\lambda}{4!}\int d^4y \langle \phi_1 \phi_2 \phi^4(y)\rangle}_{-B}\right)$$

$$\langle \phi_1 \phi_2 \rangle = \frac{I_2}{I_1} = \frac{A + B\lambda + \mathcal{O}(\lambda^2)}{1 + C\lambda + \mathcal{O}(\lambda^2)} = A + (B - AC)\lambda + \mathcal{O}(\lambda^2)$$

$$= \langle \phi_1 \phi_2 \rangle_0 - \frac{i\lambda}{4!}\int d^4y \langle \phi_1 \phi_2 \phi^4(y)\rangle_0$$
$$+ \langle \phi_1 \phi_2 \rangle_0 \frac{i\lambda}{4!}\int d^4y \langle \phi^4(y)\rangle_0 + \mathcal{O}(\lambda^2)$$

$\langle \phi_1 \phi_2 \rangle_0 = D_F(x_1 - x_2)$ Feynman propagator of free theory

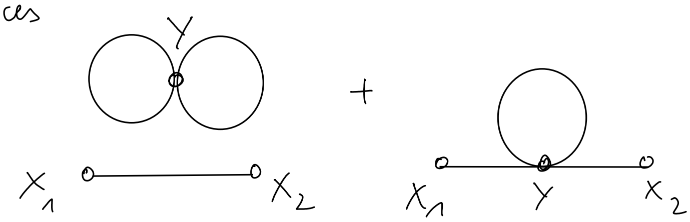
$$\langle \phi^4(y)\rangle_0 = \langle \phi\phi\phi\phi \rangle + \langle \phi\phi\phi\phi \rangle + \langle \phi\phi\phi\phi \rangle$$
$$= 3\, D_F(y-y)^2 \quad \text{by Wick's theorem}$$

$$\langle \phi_1 \phi_2 \, \phi^4(y) \rangle_0 = \dots = 3\, D_F(x_1 - x_2)\, D_F^2(y-y) \; +$$

Wick's theorem $\quad 12\, D_F(x_1 - y)\, D_F(x_2 - y)\, D_F(y-y)$

$\phi_1 \, \phi_2 \, \phi \cdots \phi$ — 4 possible choices, 3 possible choices $\;\nearrow = 4\cdot 3$

$\hat{=}$




$$\langle \phi_1 \, \phi_2 \rangle = D_F(x_1 - x_2) + \frac{1}{2} \int d^4y \, D_F(x_1 - y)\, D_F(x_2 - y)\, D_F(y-y)$$

unconnected diagram drops out

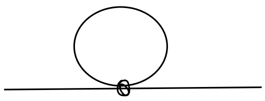
We can recover this result from Feynman rules using the symmetry factor

$$\boxed{S = \left(\frac{1}{4!}\right)^V \frac{C}{V!}}$$

V = number of vertices

C = number of contractions

i.e. for

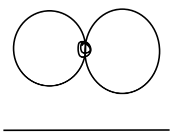


$V = 1$

$C = 4\cdot 3 = 12$

$S = \frac{12}{4!} = \frac{1}{2}$ //

or



$V = 1$

$C = 3$

$S = \frac{3}{4!} = \frac{1}{6}$ //

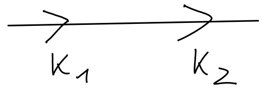
## In momentum space

$$\int dx_1^4 \, dx_2^4 \; e^{i(k_1 x_1 - k_2 x_2)} \, D_F(x_1 - x_2)$$

$$= \int d^4x_1 \, d^4x_2 \, \frac{d^4p}{(2\pi)^4} \, \frac{1}{p^2 + i\varepsilon} \, e^{i[x_1(k_1 - p) + x_2(p - k_2)]}$$

$$= (2\pi)^4 \int d^4p \; \delta(k_1 - p)\, \delta(p - k_2) \, \frac{1}{p^2 + i\varepsilon}$$

$$= (2\pi)^4 \, \delta(k_1 - k_2) \, \frac{1}{k_1^2 + i\varepsilon} \quad // \quad =$$



## Momentum conservation

$$\int d^4x_1\, d^4x_2\, d^4y \; D_F(x_1-y)\, D_F(x_2-y)\, D_F(y-y)\, e^{i(x_1 k_1 - x_2 k_2)}$$

$$= \int d^4x_1\, d^4x_2\, d^4y \frac{d^4p_1\, d^4p_2\, d^4p_3}{(2\pi)^{4\cdot 3}} \quad \frac{1}{p_1^2+i\varepsilon} \frac{1}{p_2^2+i\varepsilon} \frac{1}{p_3^2+i\varepsilon} \cdot$$

$$\exp\left[ x_1\cdot(k_1-p_1) - x_2\cdot(k_2+p_2) + y(p_1+p_2) \right]$$

$$= \int d^4p_1\, d^4p_2\, d^4p_3 \quad \delta(k_1-p_1)\, \delta(k_2+p_2)\, \delta(p_1+p_2) \cdot$$

$$\left[(p_1^2+i\varepsilon)(p_2^2+i\varepsilon)(p_3^2+i\varepsilon)\right]^{-1}$$

$\hookrightarrow$ $k_1 = p_1 \qquad k_2 = -p_2$

$$= \delta(k_1-k_2) \int d^4p \frac{1}{(k_1^2+i\varepsilon)^2(k_2^2+i\varepsilon)(p^2+i\varepsilon)}$$

Conclusion: Knowing the Feynman rules in momentum space from the lecture, we can immediately write down the correlator $\langle \phi_1 \phi_2 \rangle$ up to order $\mathcal{O}(\lambda^2)$.