Quantum Field Theory, Summer 2024

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8. One Loop Corrections (27 points)

To be discussed on Friday, 26th April, 2024 in the tutorial.
Please indicate your preferences until Sunday, 21/04/2024, 21:00:00 on the website.

We have learned now much more about loop corrections in quantum field theory. Therefore, we can revisit the question from exercise 5 about corrections to the mass of a scalar field. Remember that we studied ϕ^4-theory back then. To keep things as simple as possible, we now turn to ϕ^3 theory. It will give us the chance to revisit all ideas from the lecture without the additional complication of having to deal with spinors. We also get a glimpse at a new regularisation method, namely dimensional regularisation.

Exercise 8.1: One-loop renormalisation of ϕ^3 theory

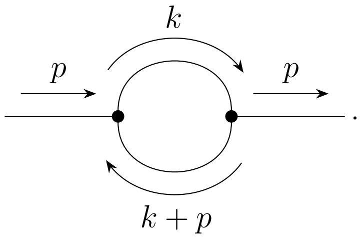
The Lagrangian we consider reads

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{m^2}{2}\phi^2}_{\mathcal{L}_0} + \underbrace{\frac{g}{3!}\phi^3}_{\mathcal{L}_I}\,, \tag{1}$$

where $\mathcal{L}_0$ is the free part of the Lagrangian and $\mathcal{L}_I$ captures interactions. From exercise 5, we already know that the propagator of the free theory is corrected to

$$\Delta(p) = \frac{1}{m^2 - p^2 - \Pi(p^2)}\,.$$

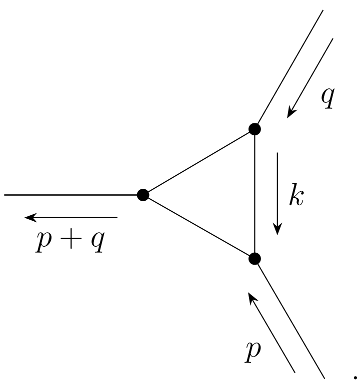
We learned that $\Pi(p^2)$ is the self-energy and that its first contribution is given by the one-loop diagram



Through a similar mechanism the three point interaction is also corrected by

$$\Gamma(p,q) = g\,(1 + \Lambda(p,q)) \tag{2}$$

with the following one-loop contribution to $\Lambda(p,q)$



a) (3 points) Write down the Feynman rules for the ϕ^3-theory given by the Lagrangian (1). Use them to show that we obtain

$$\Pi(p^2) = \frac{g^2}{2}\int \frac{\mathrm{d}^D k}{(2\pi)^D i}\frac{1}{m^2-k^2}\frac{1}{m^2-(k+p)^2} \quad \text{for the self-energy and} \tag{3}$$

$$\Lambda(p,q) = g^2\int \frac{\mathrm{d}^D k}{(2\pi)^D i}\frac{1}{m^2-k^2}\frac{1}{m^2-(k+p)^2}\frac{1}{m^2-(k-q)^2} \quad \text{for the vertex correction}\,. \tag{4}$$

Note: We perform here the integral in D dimensions. This will help us to regularise the integrals later.

b) (3 points) Introduce Feynman parameters for the self-energy (3) and perform the momentum integral.
Hint: As we have seen in the lecture, it is convenient to introduce a new momentum $l = k + xp$ and write the integral in terms of this parameter. After this is done, do a Wick rotation ($l^0 = il_E^0$, $l^2 = -l_E^2$) and use the integral

$$\int \frac{\mathrm{d}^D l_E}{(2\pi)^D}\frac{1}{(l_E^2+X)^\xi} = (4\pi)^{-\frac{D}{2}}\frac{\Gamma(\xi-\frac{D}{2})}{\Gamma(\xi)}X^{\frac{D}{2}-\xi}\,.$$

We will discuss the origin of this integral more in the next lecture while talking about dimensional regularisation.

c) (3 points) Expand the result around $\epsilon = 0$ with $\epsilon = (6-D)/2$.
Hint: Your starting point is

$$\Pi(p^2) = \frac{g^2}{2(4\pi)^{D/2}}\Gamma\left(2-\frac{D}{2}\right)\int_0^1 \mathrm{d}x\,\left[m^2 - x(1-x)p^2\right]^{\frac{D}{2}-2}\,.$$

You may also find useful the following expansion of the Gamma function near $\epsilon = 0$,

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon),$$

where γ is the Euler-Mascheroni constant.
Insert the result of the expansion in the inverse propagator to eventually find

$$\begin{aligned}
\Delta^{-1}(p^2) &= \Delta_0^{-1}(p^2) - \Pi(p^2)\\
&= m^2\left[1+\frac{g_0^2}{2(4\pi)^3}\left(\frac{1}{\epsilon}-\gamma+\ln 4\pi+1\right)\right]\\
&\quad - p^2\left[1+\frac{g_0^2}{12(4\pi)^3}\left(\frac{1}{\epsilon}-\gamma+\ln 4\pi+1\right)\right]\\
&\quad - \frac{g_0^2}{2(4\pi)^3}\int_0^1 \mathrm{d}x\,[m^2-x(1-x)p^2]\ln\frac{m^2-x(1-x)p^2}{\mu^2}\,.
\end{aligned}$$

In this equation we made the dimensionality of g explicit by introducing $g = g_0\mu^\epsilon$ such that g_0 is dimensionless. Moreover, $\Delta_0(p^2) = 1/(m^2-p^2)$ is the propagator of the free theory. Note that the divergences for $\epsilon \to 0$ only appear in the coefficients of m^2 and p^2, which suggests that we can deal with them by modifying the part of the Lagrangian that generates $\Delta_0(p^2)$.

d) (3 points) Therefore in $\mathcal{L}_0$, replace ϕ and the m by the *renormalised* quantities ϕ_r and m_r according to

$$m^2\phi^2 = m_r^2\phi_r^2(1+A) \qquad \text{and} \quad (\partial^\mu\phi)(\partial_\mu\phi) = (\partial^\mu\phi_r)(\partial_\mu\phi_r)(1+B)\,.$$

How do A and B enter into $\Delta_0(p^2)$ and $\Delta(p^2)$?

e) (3 points) Choose A and B such that they absorb the divergences. Note that there are arbitrarily many possibilities. They are related to different renormalisation schemes.

f) (3 points) Repeat the steps of b) for the vertex correction (4). *Hint you will need to introduce the Feynman parameters*

$$\frac{1}{ABC} = 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{1}{[xA + yB + (1-x-y)C]^3}\,.$$

g) (3 points) Repeat the steps of c) for the vertex correction (4). *Hint your starting point is*

$$\Lambda(p,q) = \frac{g^2}{(4\pi)^{\frac{D}{2}}}\Gamma\left(3-\frac{D}{2}\right)\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y X^{\frac{D}{2}-3}$$

with $X = m^2 - x(1-x)p^2 - y(1-y)q^2 - 2xyp\cdot q$. Insert the result into the expression for the vertex function (2) and verify that you find

$$\Gamma(p,q) = g_0\left[1 + \frac{g_0^2}{2(4\pi)^3}\left(\frac{1}{\epsilon} - \gamma + \ln 4\pi\right)\right] - \frac{g_0^3}{(4\pi)^3}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \ln\frac{K}{\mu^2}$$

with

$$K = m^2 - x(1-x)q^2 - y(1-y)p^2 - 2xyp\cdot q\,.$$

h) (3 points) Redefine the interaction term $\mathcal{L}_I$ similar to what you did in d) by introducing a third renormalisation quantity C and choose it such that the divergences are absorbed.

i) (3 points) Usually one does not modify whole terms in the Lagrangian like we did above, but rather fields, masses and couplings individually:

$$\phi = \sqrt{Z_\phi}\phi_r\,, \quad m = \sqrt{Z_m}m_r\,, \quad \text{and} \quad g = Z_g g_r\,.$$

Relate the Z_i to A, B, and C, neglecting terms of $\mathcal{O}(g^3)$.