



8. One Loop Corrections (27 points)

To be discussed on Tuesday, 9th May, 2023 in the tutorial.

Please indicate your preferences until Thursday, 04/05/2023, 21:00:00 on the website.

We have learned now much more about loop corrections in quantum field theory. Therefore, we can revisit the question from exercise 5 about corrections to the mass of a scalar field. Remember that we studied ϕ^4 -theory back then. To keep things as simple as possible, we turn now to ϕ^3 theory. It will give us the chance to revisit all ideas from the lecture without the additional complication of having to deal with spinors. We also get a glimpse at a new regularisation method, namely dimensional regularisation.

Exercise 8.1: One-loop renormalisation of ϕ^3 theory

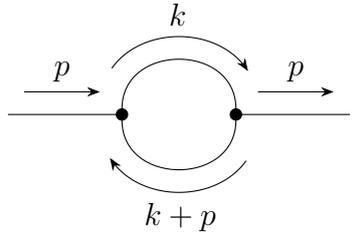
The Lagrangian we consider reads

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{m^2}{2}\phi^2}_{\mathcal{L}_0} + \underbrace{\frac{g}{3!}\phi^3}_{\mathcal{L}_I}, \quad (1)$$

where \mathcal{L}_0 is the free part of the Lagrangian and \mathcal{L}_I captures interactions. From exercise 5, we already know that the propagator of the free theory is corrected to

$$\Delta(p) = \frac{1}{m^2 - p^2 - \Pi(p^2)}.$$

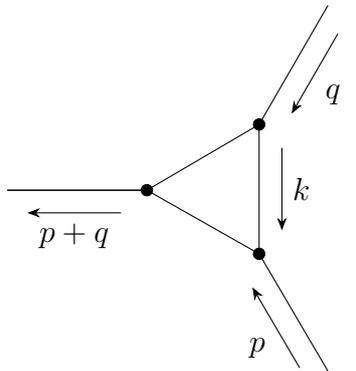
We learned that $\Pi(p^2)$ is the self-energy and that its first contribution is given by the one-loop diagram



Through a similar mechanism three point interaction are also corrected by

$$\Gamma(p, q) = g(1 + \Lambda(p, q)) \quad (2)$$

with the one-loop contribution to $\Lambda(p, q)$



- a) (3 points) Write down the Feynman rules for the ϕ^3 -theory given by the Lagrangian (1). Use them to show that we obtain

$$\Pi(p^2) = \frac{g^2}{2} \int \frac{d^D k}{(2\pi)^D i} \frac{1}{m^2 - k^2} \frac{1}{m^2 - (k+p)^2} \quad \text{for the self-energy and} \quad (3)$$

$$\Lambda(p, q) = g^2 \int \frac{d^D k}{(2\pi)^D i} \frac{1}{m^2 - k^2} \frac{1}{m^2 - (k+p)^2} \frac{1}{m^2 - (k-q)^2} \quad \text{for the vertex-correction.} \quad (4)$$

Note: We perform here the integral in D dimensions. This will help us to regularise the integrals later.

- b) (3 points) Introduce Feynman parameters for the self-energy (3) and perform the momentum integral.

Hint: As we have seen in the lecture, it is convenient to introduce a new momentum $l = k + xp$ and write the integral in terms of this parameter. After this is done, do a Wick rotation ($l^0 = il_E^0$, $l^2 = -l_E^2$) and use the integral

$$\int \frac{d^D l_E}{(2\pi)^D} \frac{1}{(l_E^2 + X)^\xi} = (4\pi)^{-\frac{D}{2}} \frac{\Gamma(\xi - \frac{D}{2})}{\Gamma(\xi)} X^{\frac{D}{2} - \xi}.$$

We will discuss the origin of this integral more in the next lecture while talking about dimensional regularisation.

- c) (3 points) Expand the result around $\epsilon = 0$ with $\epsilon = (6 - D)/2$.

Hint: Your starting point is

$$\Pi(p^2) = \frac{g^2}{2(4\pi)^{D/2}} \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 dx [m^2 - x(1-x)p^2]^{\frac{D}{2} - 2}.$$

Insert the result in the inverse propagator to eventually find

$$\begin{aligned} \Delta^{-1}(p^2) &= \Delta_0^{-1}(p^2) - \Pi(p^2) \\ &= m^2 \left[1 + \frac{g_0^2}{2(4\pi)^3} \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi + 1 \right) \right] \\ &\quad - p^2 \left[1 + \frac{g_0^2}{12(4\pi)^3} \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi + 1 \right) \right] \\ &\quad - \frac{g^2}{2(4\pi)^3} \int_0^1 dx [m^2 - x(1-x)p^2] \ln \frac{m^2 - x(1-x)p^2}{\mu^2}. \end{aligned}$$

In this equation we made the dimensionality of g explicit by introducing $g = g_0 \mu^\epsilon$ such that μ is dimensionless. Moreover, $\Delta^0(p^2) = 1/(m^2 - p^2)$ is the propagator of the free theory. Note that the divergences for $\epsilon \rightarrow 0$ only appear in the coefficients of m^2 and p^2 , which suggest that we can deal with them by modifying the part of the Lagrangian that generates $\Delta_0(p^2)$.

- d) (3 points) Therefore replace in \mathcal{L}_0 ϕ and the m by the *renormalised* quantities ϕ_r and m_r according to

$$m^2 \phi^2 = m_r^2 \phi_r^2 (1 + A) \quad \text{and} \quad (\partial^\mu \phi)(\partial_\mu \phi) = (\partial^\mu \phi_r)(\partial_\mu \phi_r) (1 + B).$$

How do A and B enter into $\Delta_0(p^2)$ and $\Delta(p^2)$?

- e) (3 points) Choose A and B such that they absorb the divergences. Note that there are arbitrarily many possibilities. They are related to different renormalisation schemes.
- f) (3 points) Repeat the steps of b) for the vertex correction (4). *Hint you will need to introduce the Feynman parameters*

$$\frac{1}{ABC} = 2 \int_0^1 dx \int_0^{1-x} dy \frac{1}{[xA + yB + (1-x-y)C]^3}.$$

- g) (3 points) Repeat the steps of c) for the vertex correction (4). *Hint your starting point is*

$$\Lambda(p, q) = \frac{g^2}{(4\pi)^{\frac{D}{2}}} \Gamma\left(3 - \frac{D}{2}\right) \int_0^1 dx \int_0^{(1-x)} dy X^{\frac{D}{2}-3}$$

with $X = m^2 - x(1-x)p^2 - y(1-y)q^2 - 2xypq$. Insert the result into the expression for the vertex function (2) and verify that you find

$$\Gamma(p, q) = g_0 \left[1 + \frac{g_0^2}{2(4\pi)^3} \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi \right) \right] - \frac{g_0^3}{(4\pi)^3} \int_0^1 dx \int_0^{1-x} dy \ln \frac{K}{\mu^2}$$

with

$$K = m^2 - x(1-x)q^2 - y(1-y)p^2 - 2xyp \cdot q.$$

- h) (3 points) Redefine the interaction term \mathcal{L}_I similar to what you did in d) by introducing a third renormalisation quantity C and choose it such that the divergences are absorbed.
- i) (3 points) Usually one does not modify whole terms in the Lagrangian like we did above, but rather fields, masses and couplings individually:

$$\phi = \sqrt{Z_\phi} \phi_r, \quad m = \sqrt{Z_m} m_r, \quad \text{and} \quad g = \sqrt{Z_g} g_r.$$

Relate the Z_i to A , B , and C , neglecting terms of $\mathcal{O}(g^3)$.