Gauge/Gravity Duality, Winter 2024/25

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7. Conformal Field Theory (17 points)

To be discussed on Friday, 29th November, 2024 in the tutorial.
Please indicate your preferences until Sunday, 24/11/2024, 21:00:00 on the website.

Exercise 7.1: Conformal transformations.

The conformal algebra in $d > 2$ dimensions is given by

$$\begin{aligned}
[J_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \\
[D, P_\mu] &= iP_\mu, \\
[D, K_\mu] &= -iK_\mu, \\
[D, J_{\mu\nu}] &= 0, \\
[K_\mu, K_\rho] &= 0, \\
[K_\mu, P_\nu] &= -2i(\eta_{\mu\nu}D - J_{\mu\nu}).
\end{aligned}$$

a) (1 point) Show that under the identifications

$$\begin{aligned}
\bar{J}_{d(d+1)} &= -D, \\
\bar{J}_{\mu d} &= \frac{1}{2}(K_\mu - P_\mu), \\
\bar{J}_{\mu(d+1)} &= \frac{1}{2}(K_\mu + P_\mu),
\end{aligned}$$

the conformal algebra can be written in terms of an $\mathfrak{so}(d, 2)$.

b) (1 point) The finite special conformal transformations are defined by

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2}.$$

Show that special conformal transformations can be decomposed into an inversion $x^\mu \to x'^\mu = \frac{x^\mu}{x^2}$, a translation $x'^\mu \to x''^\mu = x'^\mu + b^\mu$ and another inversion $x''^\mu \to x'''^\mu = \frac{x''^\mu}{x''^2}$.

c) (3 points) Consider $d = 2$. Recalling the definition of the generators $l_n = -z^{n+1}\partial_z$, $\bar{l}_m = -\bar{z}^{n+1}\partial_{\bar{z}}$ from the lecture, show explicitly that they satisfy the following algebra

$$[l_n, l_m] = (m - n)l_{m+n}, \qquad [\bar{l}_n, \bar{l}_m] = (m - n)\bar{l}_{m+n}, \qquad [l_n, \bar{l}_m] = 0.$$

Exercise 7.2: Transformations of fields.

a) (4 points) Use a general infinitesimal conformal transformation

$$\epsilon^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2(b \cdot x)x^\mu$$

to show that, for a primary field $\phi(x)$,

$$\begin{aligned}
[P_\mu, \phi(x)] &= -i\partial_\mu \phi(x) := \mathcal{P}_\mu \phi(x), \\
[D, \phi(x)] &= -i\Delta\phi(x) - ix^\mu \partial_\mu \phi(x) := \mathcal{D}\phi(x), \\
[J_{\mu\nu}, \phi(x)] &= -\mathcal{J}_{\mu\nu}\phi(x) + i(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi(x) := \tilde{\mathcal{J}}\phi(x), \\
[K_\mu, \phi(x)] &= \left[i(-x^2\partial_\mu + 2x_\mu x^\rho \partial_\rho + 2x_\mu\Delta) - 2x^\nu \mathcal{J}_{\mu\nu}\right]\phi(x) := \mathcal{K}_\mu \phi(x),
\end{aligned}$$

may be synthetised in the form

$$\delta_\epsilon \phi(x) = -\mathcal{L}_\nu \phi(x), \qquad \mathcal{L}_\nu = \epsilon(x)\cdot\partial + \frac{\Delta}{d}\partial\cdot\epsilon(x) - \frac{i}{2}\partial_{[\mu}\epsilon_{\nu]}(x)\mathcal{J}^{\mu\nu}.$$

b) (4 points) Show that $\mathcal{P}_\mu, \mathcal{D}, \tilde{\mathcal{J}}, \mathcal{K}_\mu$ satisfy the conformal algebra of exercise 7.1, i.e., they form a representation of that algebra.

Exercise 7.3: Stress-energy tensor.

In previous lectures we were defining the energy-momentum tensor as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}}{\delta g^{\mu\nu}}.$$

a) (4 points) Show that $T_{\mu\nu}$ is traceless if the action $\mathcal{S}$ is scale invariant.