Quantum Field Theory, Summer 2023

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# 6. Fermionic Path Integral (18 points)

To be discussed on Tuesday, 18th April, 2023 in the tutorial.
Please indicate your preferences until Thursday, 13/04/2023, 21:00:00 on the website.

During the lecture, I promised that we will have a detailed look at how to calculate fermionic partition functions. To make it a bit more interesting, we combine this technical aspect with the concept of symmetries and encounter a very interesting phenomena, the chiral anomaly.

**Exercise 6.1: Chiral Anomaly**

Last lecture, we mentioned that the measure in the path integrals requires some extra care. This is in particular important when we study the fate of classical symmetries under quantum corrections. They may break down and lead to anomalies. As an example, we will study here the chiral anomaly. More precisely, we will study a massless Dirac field coupled to a non-dynamic U(1) gauge field (like in exercise 2.3 for $m = 0$). With non-dynamic we just indicate that the action does not contain the kinetic term $F_{\mu\nu}F^{\mu\nu}$,

$$S = \int \mathrm{d}^4x\, \overline{\psi} \not{D} \psi \,. \tag{1}$$

a) (3 points) Check that the action (1) is invariant under axial symmetry

$$\psi' = e^{i\gamma_4 \alpha(x)}\psi \,, \qquad \gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3 \,.$$

Use the covariant derivative $\not{D} = \not{\partial} + i\not{A}$. How does $A_\mu$ transform?

b) (3 points) Show how this symmetry gives rise to the chiral current

$$j^\mu = \overline{\psi}\gamma^\mu\gamma_4\psi$$

and that this current is governed by the conservation law $0 = \partial_\mu j^\mu$ after imposing the field equations.

c) (3 points) We now want to study the fate of this conservation law in the path integral. Like we did already several times before, we first expand the field $\psi$ in eigenfunctions $\psi_i$ of the Dirac operator. They are formally defined by

$$i\not{D}\psi_i = \lambda_i\psi_i \,,$$

and give rise to

$$\psi = \sum_i a_i\psi_i \quad \text{and} \quad \overline{\psi} = \sum_i \bar{b}_i\psi_i^\dagger \,. \tag{2}$$

Here $a_i$ and $\bar{b}_i$ are mutual anti-commuting Grassmann variables. Prove that the $\psi_i$'s can be normalised such that

$$\langle \psi_i | \psi_j \rangle = \int \mathrm{d}^4x \psi_i^\dagger(x)\psi_j(x) = \delta_{ij} \,.$$

Finally, show that the path integral measure $\mathcal{D}\overline{\psi}\mathcal{D}\psi$ can be written as $\int \prod_i \mathrm{d}a_i \, \mathrm{d}\bar{b}_i$. Note that this expression assumes a certain discrimination, similar to the one we discussed in the lecture. However the details are not important for what we want to show. Thus, we keep it at the formal level.

d) (3 points) Compute the transformation matrix $C_{ij}$ that mediates the infinitesimal version of the axial symmetry discovered above, namely

$$\psi' = [1 + i\alpha(x)\gamma_4]\,\psi \quad \leftrightarrow \quad a'_i = \sum_j C_{ij} a_j\,.$$

How does $\overline{b}_i$ transform?

e) (3 points) Using the results from c) and d), show that the measure actually transforms as

$$\prod_i \mathrm{d}a_i\,\mathrm{d}\overline{b}_i = \prod_i \mathrm{d}a'_i\,\mathrm{d}\overline{b}'_i \exp\left[-2i\alpha(x)\mathcal{A}(x)\right]$$

with

$$\mathcal{A}(x) = \sum_i \psi_i^\dagger(x)\gamma_4\psi_i(x)\,.$$

Now, please demonstrate that the partition function transforms

$$Z = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-iS}$$

transforms into

$$Z' = \int \mathcal{D}\overline{\psi}'\mathcal{D}\psi' \exp\left[-iS - \int \mathrm{d}^4x\,\alpha(x)\left(\partial_\mu j^\mu - 2i\mathcal{A}(x)\right)\right]\,.$$

Therefore, the classical conservation law gets modified to

$$\partial_\mu j^\mu = -2i\mathcal{A}(x)\,.$$

This is called the chiral or axial anomaly.

f) (3 points) The integral $\int \mathrm{d}^4x\,\mathcal{A}(x)$ diverges and has to be regularised. Following Fujikawa's method, we use heat kernel regularisation. Explain how this works by looking at a textbook where this procedure is explain (section 13.2.1 of "Geometry, Topology and Physics" written by Mikio Nakahara is great place to start). In particular demonstrate that eventually, we obtain

$$\partial_\mu j^\mu = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$$

with the abelian field strength $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$.