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5. Roots and Cartan-Weyl basis (14 points)

To be discussed on Monday, 2nd June, 2025 in the tutorial.
Please indicate your preferences until Wednesday, 28/05/2025, 21:00:00 on the website.

Exercise 5.1: Roots of sl(3,C)

Consider the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ which is the complexification of $\mathfrak{su}(3)$.

a) (2 points) Show that $\mathfrak{sl}(3,\mathbb{C})$ possesses abelian subalgebras of dimension two which are not Cartan subalgebras, i.e. whose elements are generically not ad-diagonalisable.

b) (2 points) Using the following basis for its Cartan subalgebra $\mathfrak{h}$,

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{1}$$

Let E_{ij} be the matrix which has 1 in the (i,j) position and 0 elsewhere, i.e. $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Compute the coefficients α_{ij} in the following equation (where no Einstein summation is implied),

$$[H, E_{ij}] = \alpha_{ij}(H) E_{ij}, \qquad \text{for all } H \in \mathfrak{h}. \tag{2}$$

How many roots do you have?

c) (2 points) Compute $[E_{ij}, E_{kl}]$.

d) (2 points) Show that $[E_{ij}, E_{ji}]$ lies in the Cartan subalgebra $\mathfrak{h}$.

Exercise 5.2: Killing form and Cartan subalgebra

a) (3 points) Prove that the restriction of the Killing form of a semisimple Lie algebra to the Cartan subalgebra is non-degenerate.

b) (1 point) Explain why the characteristic equation for elements of the Cartan subalgebra has precisely rank $\mathfrak{g}$ 'roots' which are identically zero.

Exercise 5.3: CSA of so(4)

a) (2 points) Let $\mathfrak{h} \subset \mathfrak{so}(4,\mathbb{C})$ be the subalgebra consisting of matrices of the form

$$\begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}. \tag{3}$$

Show that $\mathfrak{h}$ is a Cartan subalgebra and find the corresponding root decomposition.