Selected Tools of Modern Theoretical Physics 2B, Summer 2025

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5. Representation theory (16 points)

To be discussed on Monday, 26th May, 2025 in the tutorial.
Please indicate your preferences until Wednesday, 21/05/2025, 21:00:00 on the website.

Exercise 5.1: Killing form

During the lecture, we defined an inner product on the Lie algebra $\mathfrak{g}$ called the *Killing form* κ. Let us study some of its properties.

a) (2 points) An *automorphism* ω of $\mathfrak{g}$ is a bijective map $\omega : \mathfrak{g} \to \mathfrak{g}$ which is compatible with the Lie bracket, i.e. it obeys

$$\omega([x, y]) = [\omega(x), \omega(y)].$$

Show that the Killing form κ is invariant under any automorphism of $\mathfrak{g}$.

b) (2 points) Prove the ad-invariance of the Killing form, i.e. the property

$$\kappa([x, y], z) = \kappa(x, [y, z]), \quad \forall\, x, y, z \in \mathfrak{g}.$$

c) (2 points) Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) = \mathrm{span}(h, e, f)$ with Lie brackets $[h, e] = 2e, [h, f] = -2f$ and $[e, f] = h$. Compute the Killing form and show that it obeys

$$\kappa(x, y) = 4\,\mathrm{Tr}(xy), \quad \forall\, x, y \in \mathfrak{sl}(2, \mathbb{R}).$$

d) (2 points) Repeat the exercise for $\mathfrak{so}(3)$. Is it semisimple?

e) (2 points) Repeat this for the three-dimensional Heisenberg algebra which we considered during the lecture. Is it semisimple?

Exercise 5.2: Low-dimensional Lie algebras

a) (2 points) Consider the two-dimensional Lie algebra with generators T and U and Lie bracket $[T, U] = U$, and the three-dimensional Heisenberg algebra. Are these Lie algebras solvable? Are they nilpotent?

b) (2 points) Classify all two-dimensional Lie algebras up to isomorphism.

Exercise 5.3: Adjoint action

a) (2 points) During the lecture, we defined the adjoint actions Ad and ad for Lie groups and Lie algebras respectively. Prove the identities

$$(\mathrm{Ad}_g)^{-1} = \mathrm{Ad}_{g^{-1}}, \quad \mathrm{ad}_{\mathrm{Ad}_g(X)} = \mathrm{Ad}_g\, \mathrm{ad}_X (\mathrm{Ad}_g)^{-1}.$$