Quantum Field Theory, Summer 2024

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4. Gaussian Integrals (18 points)

To be discussed on Friday, 22nd March, 2024 in the tutorial.
Please indicate your preferences until Sunday, 17/03/2024, 21:00:00 on the website.

Last lecture, we have encountered a new tool, the path integral. So far, we just have shown that it gives us an interesting new perspective on results we already know from the canonical quantisation. Its full power will just become obvious in the next lecture. To prepare for this revelation, we will study here some interesting properties of Gaussian integrals following the nice remark from the book Quantum Field Theory in a Nutshell by A. Zee:

Believe it or not, a significant fraction of the theoretical physics literature consists of varying and elaborating this basic Gaussian integral.

Exercise 4.1: Integrals, Integral, Integrals

a) (3 points) Do the integral

$$G = \int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{1}{2}x^2}$$

by first squaring it and then switching to polar coordinates. Now use the result to show

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+Jx} = \sqrt{\frac{2\pi}{a}} e^{J^2/(2a)}\,. \tag{1}$$

b) (3 points) We mentioned in the lecture that moments

$$\langle x^n \rangle = \frac{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\, x^n e^{-\frac{1}{2}ax^2}}{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{1}{2}ax^2}}$$

are an important tool to evaluate path integrals. Explain why $\langle x^{2n+1} \rangle = 0$ for $n \in \mathbb{N}$. Now compute $\langle x^{2n} \rangle$ by repeatedly applying $\mathrm{d}/\mathrm{d}a$ to (1) with $J = 0$. What are the combinatorics behind the prefactor you obtain?

c) (3 points) To transition from a simple integral to a path integral, substitute x in the above expressions by the vector x_i, A by the matrix A^{ij} and J by the vector J^i with $i, j = 1, \dots, N$. Prove that

$$\int_{-\infty}^{\infty} \mathrm{d}x^1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x^N e^{-\frac{1}{2}x_i A^{ij} x_j + J^i x_i} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{J^i (A^{-1})_{ij} J^j/2} \tag{2}$$

holds (Einstein sum convention is implied).

d) (3 points) Repeat the steps from problem b to compute first $\langle x_i x_j \rangle$ and then find a convenient way to compute $\langle x_i x_j \dots x_k x_l \rangle$.

Hint: What you are actually doing is directly following from Wick's theorem. Try computing $\langle x_i x_j x_k x_l \rangle$ and from this you should be able to generalise.

Exercise 4.2: And a Little Bit of Physics

a) (3 points) Compute the value of A_{ij} which enters the path integral (2) for the discretised action

$$S = \sum_{k=1}^{N} \left[\frac{m}{2} \frac{(x_{k+1} - x_k)^2}{\epsilon} - \epsilon\lambda \left(\frac{x_{k+1} + x_k}{2} \right)^2 \right] ,$$

we discussed in the lecture. Compute the correlator $\langle x_i x_j \rangle = (A^{-1})_{ij}$. What is the interpretation of λ?

b) (3 points) Show how the integration $\int \mathcal{D}\Pi$ in

$$\int \mathcal{D}\phi \, \mathcal{D}\Pi \exp \left[i \int_0^T \mathrm{d}^4 x \, L \right]$$

can be computed if the Lagrangian is quadratic in Π.