Selected Tools of Modern Theoretical Physics 2B, Summer 2025

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4. Exponential map (15 points)

To be discussed on Monday, 19th May, 2025 in the tutorial.
Please indicate your preferences until Wednesday, 14/05/2025, 21:00:00 on the website.

Exercise 4.1: Maurer-Cartan equation and Jacobi identity

During the lecture, we saw that left-invariant one-forms on a Lie group G satisfy the *Maurer-Cartan formula*:

$$\mathrm{d}e^a + \frac{1}{2} f_{bc}{}^a e^b \wedge e^c = 0. \tag{1}$$

a) (2 points) Show that nilpotency of the exterior derivative $\mathrm{d}^2 = 0$ and the above formula imply the following identity for the structure constants of $\mathfrak{g}$,

$$f_{d[a}{}^f f_{bc]}{}^d = 0. \tag{2}$$

b) (2 points) Show how the above equation (2) is also a consequence of the Jacobi identity of $\mathfrak{g}$.

Exercise 4.2: Exponential map

a) (2 points) Let $G = SO(2)$. Show that the exponential map reaches all elements of G, and describe the one-parameter subgroup generated by

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

b) (2 points) Express the exponentials $\exp(\xi M_+)$ and $\exp(\xi M_-)$ of the matrices

$$M_\pm = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}, \tag{4}$$

in terms of trigonometric and hyperbolic functions. Which range of values of the variable ξ is needed to describe these exponentials?

c) (2 points) Consider $G = SL(2, \mathbb{R})$ and its Lie algebra elements

$$M = a_A T^A = \begin{pmatrix} a_0 & a_+ \\ a_- & -a_0 \end{pmatrix}.$$

The square is

$$M^2 = X I_{2\times 2}, \quad \text{where } X = a_0^2 + a_+ a_-.$$

Prove that the exponential map is

- For $X > 0$:

$$\exp(M) = \cosh(\sqrt{X}) I + \frac{M \sinh(\sqrt{X})}{\sqrt{X}} = \begin{pmatrix} \frac{a_0 \sinh(\sqrt{X})}{\sqrt{X}} + \cosh(\sqrt{X}) & \frac{a_+ \sinh(\sqrt{X})}{\sqrt{X}} \\ \frac{a_- \sinh(\sqrt{X})}{\sqrt{X}} & -\frac{a_0 \sinh(\sqrt{X})}{\sqrt{X}} + \cosh(\sqrt{X}) \end{pmatrix}.$$

- For $X < 0$:

$$\exp(M) = \cos(\sqrt{-X})I + \frac{M\sin(\sqrt{-X})}{\sqrt{-X}} = \begin{pmatrix} \frac{a_0\sin(\sqrt{-X})}{\sqrt{-X}} + \cos(\sqrt{-X}) & \frac{a_+\sin(\sqrt{-X})}{\sqrt{-X}} \\ \frac{a_-\sin(\sqrt{-X})}{\sqrt{-X}} & -\frac{a_0\sin(\sqrt{-X})}{\sqrt{-X}} + \cos(\sqrt{-X}) \end{pmatrix}.$$

d) (2 points) Check that in each case this gives real matrices that are indeed in $SL(2,\mathbb{R})$, and check the continuity around $X = 0$.

e) (2 points) Consider the matrices

$$A(\alpha) = \begin{pmatrix} -1 & \alpha \\ 0 & -1 \end{pmatrix}, \quad \alpha \in \mathbb{R}, \tag{5}$$

which are also in $SL(2,\mathbb{R})$. Prove that this is in the image of the exponential map only for $\alpha = 0$. The space where $\alpha \neq 0$ is thus a "corner of the non-compact group".

f) (1 point) Show that the matrices close to the identity, i.e. $B(\alpha) = -A(\alpha)$, are always in the image of the exponential map $\forall\alpha \in \mathbb{R}$.