Selected Tools of Modern Theoretical Physics 2B, Summer 2025

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3. Differential geometry on Lie groups (16 points)

To be discussed on Monday, 12th May, 2025 in the tutorial.
Please indicate your preferences until Wednesday, 07/05/2025, 21:00:00 on the website.

Exercise 3.1: Left-invariant tensor fields on a Lie group

Consider any group G (not necessarily Lie). Given any element $g \in G$, let us define the map

$$L_g : G \to G, \quad h \mapsto L_g h := gh \quad \text{(left multiplication)}\,.$$

A tensor field T of type (p, q) on G is said to be left-invariant if it satisfies

$$L_g^* T = T \qquad \text{for all } g \in G,$$

where L_g^* is the pull-back of the map L_g defined above. Show that

a) (1 point) the left-invariant functions (tensor fields of type $(0, 0)$) are constant functions on G.

b) (2 points) the left-invariant tensor fields are uniquely specified by its value at the unit element $e \in G$ of the group (or at any other point $h \in G$).

Exercise 3.2: Frame fields and parallelizability

Let E_a be a basis of the tangent space in the unit element, and let e_a denote the left-invariant vector fields generated by E_a, i.e.

$$e_a(g) = (L_g)_* E_a, \qquad E_a = e_a(e).$$

Show that

a) (1 point) the fields e_a constitute a global frame field on G, i.e. there exists $n = \dim G$ nowhere vanishing vector fields, being moreover linearly independent at each point.

b) (1 point) any Lie group is a parallelizable (i.e. it has a global frame field) and orientable manifold (i.e. there exists a global volume form).

c) (1 point) the vector field $V = V^a e_a$ is left-invariant if and only if it has constant components V^a with respect to the left-invariant frame field e_a.

d) (1 point) if $\hat{e}_a = A_a^b e_b$ is any other left-invariant frame field, then the transition matrix A_a^b is necessarily constant.

Exercise 3.3: Coframe fields

Let E^a be the basis of the cotangent space which is dual to the basis E_a, i.e. $\langle E^a, E_b \rangle = \delta_b^a$, $e^a(g) = (L_g)_* E^a$, and $E^a = e^a(e)$. Show that

a) (2 points) e^a and e_a are also dual to each other, i.e. at every point $g \in G$

$$\langle e^a(g), e_b(g) \rangle = \delta^a_b \,.$$

Exercise 3.4: General linear group

Find all left-invariant one-forms of the group $GL(n, \mathbb{R})$. In order to do this, let us proceed step by step:

a) (1 point) Show that in coordinates x^i_j (i.e. the matrix elements) for $x \in GL(n, \mathbb{R})$, the left multiplication by $A \in GL(n, \mathbb{R})$ reads

$$x^i_j \mapsto (L_A x)^i_j = (Ax)^i_j = A^i_k x^k_j \,.$$

b) (2 points) A generic one-form α on $GL(n, \mathbb{R})$ is of the form

$$\alpha = a^i_j(x) \mathrm{d}x^j_i \equiv \mathrm{Tr}(a(x)\mathrm{d}x) \,.$$

Show that the condition of left-invariance of α, i.e. $L^*_A \alpha = \alpha$, leads us to the requirement

$$a^i_k(Ax) A^k_j = a^i_j(x) \,. \tag{1}$$

c) (1 point) Show that the most general solution to (1) is

$$a^i_j(x) = (Cx^{-1})^i_j \equiv C^i_k (x^{-1})^k_j \,,$$

where $C \equiv a(I_n)$ is an arbitrary constant $n \times n$ matrix.

d) (1 point) Show that the most general left-invariant one-form is parametrised by a matrix C and reads

$$\alpha \equiv \alpha_C = C^i_k (x^{-1})^k_l \mathrm{d}x^l_i \equiv \mathrm{Tr}(Cx^{-1}\mathrm{d}x) \,.$$

A basis for the space of left-invariant one-forms is hence provided by computing the matrix $\theta = x^{-1}\mathrm{d}x$, which is what we call the *Maurer-Cartan* one-form of $GL(n, \mathbb{R})$.

e) (2 points) Compute the matrix $\theta = x^{-1}\mathrm{d}x$ for $x \in SO(2)$. What are the left-invariant one-form(s) and vector field(s)?