Gauge/Gravity Duality, Winter 2024/25

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4. Einstein's fields equations and AdS space (26 points)

To be discussed on Friday, 25th October, 2024 in the tutorial.
Please indicate your preferences until Sunday, 20/10/2024, 20:00:00 on the website.

Exercise 4.1: The energy-momentum tensor revisited

In the lecture, we learned that the Einstein-Hilbert action coupled to matter gives rise to the field equations

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{1}$$

where the energy momentum tensor in curved space is defined by

$$-\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}\,. \tag{2}$$

a) (2 points) Eliminate the curvature scalar R from the field equations (1)by computing and removing its trace.

b) (3 points) Use the alternative definition of the energy-momentum tensor (2) and determine $T_{\mu\nu}$ for a real scalar field ϕ with mass m and the interaction

$$L_{\text{int}}(\phi) = -\frac{g_n}{n!}\phi^n\,.$$

c) (3 points) Compute the trace of the energy-momentum tensor, if we consider a massless free theory with the additional contribution

$$S_{R\phi^2} = -\xi \int \mathrm{d}^d x \sqrt{-g} R \phi^2$$

to the Lagrangian. For which value ξ does the trace vanish?

d) (2 points) Compute the field equations for the scalar ϕ for the matter Lagrangian used in the last task and show that $S_{R\phi^2}$ induces a mass correction for the ϕ proportional to the curvature R.

e) (2 points) Show that for the value of ξ computed in task c, the action for ϕ is invariant under local *conformal transformations*

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega^{-2}(x) g_{\mu\nu}\,.$$

Exercise 4.2: Coordinates of AdS_{d+1}

We approach Anti-de Sitter space through the hypersurface

$$-L^2 = \tilde{\eta}_{MN} X^M X^N = -\left(X^{d+1}\right)^2 - \left(X^0\right)^2 + \sum_{i+1}^{d} \left(X^i\right)^2\,, \tag{3}$$

where $X \in \mathbb{R}^{d,2}$ with $\mathrm{d}s^2 = \tilde{\eta}_{MN} X^M X^N$. In the following we will use two different parameterizations. The first one, is called *global coordinates* with (ρ, τ, Ω_i),

$$\begin{aligned} X^{d+1} &= L \cosh \rho \sin \tau \\ X^0 &= L \cosh \rho \cos \tau \\ X^i &= L \sinh \rho \Omega_i \,. \end{aligned}$$

We have encountered them in the lecture and remember that $i = 1, \ldots, d$ and $\sum_{i=1}^{d} \Omega_i^2 = 1$.

a) (2 points) Compute the induced metric $\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$ in global coordinates. *Hint: Revisit the computation from the two-sphere, I presented in the 2nd lecture.*

b) (2 points) Replace ρ by $r = L \sinh \rho$ and show that the metric can be written in the form

$$\mathrm{d}s^2 = -H(r)\mathrm{d}t^2 + H(r)^{-1}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{d-1}^2 \,,$$

where $\mathrm{d}\Omega_{d-1}^2$ is the metric of the unit $(d-1)$-sphere, S^{d-1}.

c) (3 points) The *Poincaré patch coordinates* (x^μ, u), $\mu = 0, \ldots, d-1$, are defined by

$$\begin{aligned} X^{d+1} + X^d &= u \\ -X^{d+1} + X^d &= v \\ X^\mu &= \frac{u}{L} x^\mu \,. \end{aligned}$$

Using (3), eliminate v in terms of u and x^μ and show that the induced metric for (u, x^μ) takes the compact form

$$\mathrm{d}s^2 = L^2 \frac{\mathrm{d}u^2}{u^2} + \frac{u^2}{L^2} \mathrm{d}x^\mu \mathrm{d}x_\mu \,.$$

Finally, introduce $z = \frac{L^2}{u}$ and show that the metric takes the very simple form

$$\mathrm{d}s^2 = \frac{L^2}{z^2} \left(\mathrm{d}z^2 + \mathrm{d}x^\mu \mathrm{d}x_\mu \right) \,. \tag{4}$$

d) (2 points) Redraw the embedding of AdS_2, in $\mathbb{R}^{1,2}$ presented in the lecture. Which part of this AdS spacetime is not covered by Poincaré coordinates? *Hint: z only takes positive values. Why?*

Exercise 4.3: Curvature of AdS and the cosmological constant

In the last problem we have found a very simple form from the metric of Anti-de Sitter spacetime in Poicaré coordinates given in (4).

a) (3 points) Compute the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R for this metric.

b) (2 points) Show with the results from the last task that the AdS spacetime solves the vacuum Einstein field equations (1) (with vanishing energy-momentum). Determine the required cosmological constant.