



### 3. The Eightfold Way (15 points)

To be discussed on Friday, 15<sup>th</sup> March, 2024 in the tutorial.

Please indicate your preferences until Sunday, 10/03/2024, 21:00:00 on the website.

During the lecture we got puzzled about how to identify pions from the isospin triplet. In this exercise we want to look at this problem and, while doing so, learn more about the Lie algebra of the Lie group  $SU(3)$ .

#### Exercise 3.1: A Primer in Lie Groups and Lie Algebras

Particularly important for our discussion is the Lie group  $SU(3)$  and its Lie algebra. It consists of all unitary  $3 \times 3$  matrices  $U$  with determinant one.

- a) (3 points) Like in the lecture for  $SU(2)$ , we want to write them as

$$U(\alpha^J) = e^{i\alpha^J \lambda_J} \quad (1)$$

with  $J = 1, \dots, 8$  and eight linearly independent, hermitian matrices  $\lambda_J$ .

Show that  $\lambda_I^\dagger = \lambda_I$  implies that  $U$  is unitary ( $U^\dagger = U^{-1}$ ). But, this is not enough, we require moreover that  $\det U = 1$ , too. Explain why this is only possible if all  $\lambda_I$ 's are traceless. Finally prove that all traceless hermitian  $3 \times 3$  matrices  $\lambda$  can be written as the sum  $\lambda = \alpha^J \lambda_J$  over eight different  $\lambda_J$ 's (assuming the  $\alpha^J$ 's are real).

- b) (3 points) One can pick two generators out of the eight which commute. We pick

$$H_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H_2 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (2)$$

All six remaining generators can be written as eigenvectors under the adjoint action of  $H_i$ ,  $i = 1, 2$ ,

$$[H_i, E_{\bar{\alpha}}] = \alpha_i E_{\bar{\alpha}}. \quad (3)$$

Hence, in total you should find six different tuples of eigenvalues  $(\alpha_1, \alpha_2)$ . Plot them and show that they are on a grid spanned by  $(1/2, \sqrt{3}/2)$  and  $(1/2, -\sqrt{3}/2)$ , called the *root lattice* of  $SU(3)$ .

- c) (3 points) Show that the two roots on the  $\alpha_2 = 0$  axis form an  $SU(2)$  subalgebra after an appropriate choice of a third generator. Finally, find the coefficients in  $Q = c_1 H_1 + c_2 H_2$  such that the adjoint action  $[Q, E_{\alpha}] = q_{\alpha} E_{\alpha}$  has only eigenvalues  $-1, 0$ , or  $1$ .

#### Exercise 3.2: The Physics Behind The Math

Now, we want relate this mathematical framework with some physics!

- a) (3 points) Show that the four components of the Dirac spinor  $\psi$  capture a fermion with spin  $\pm 1/2$  and that  $\bar{\psi}$  describes the corresponding anti-particle (see problems 1.c) and 1.d) from the first exercise sheet). We can now identify the spinors  $U$  acts on as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad \text{and} \quad \bar{\psi} = (\bar{\psi}_1 \quad \bar{\psi}_2 \quad \bar{\psi}_3) = (\bar{u} \quad \bar{d} \quad \bar{s}) . \quad (5)$$

What is the physical interpretation of  $u, d, s$  and their barred counterparts?

- b) (3 points) Show that the combination  $\bar{\psi}^i \psi_j$ , with  $i$  and  $j = 1, 2, 3$  is invariant under Lorentz transformations. We can therefore identify it with pseudoscalar mesons. Take all these mesons and organise them into the plot of the root diagram according to their electric charge  $Q$  and strangeness  $S = \alpha_2$ .

### Bonus Exercise 3.3: Taking Spin into Account

3 bonus points

*Warning: This is a hard problem. But it is fun, too :-). If you don't have a solid background in representation theory you might avoid it. Most likely I will present it and show you how to have some fun stuff with Mathematica if we have the time.*

As we have seen in the last problem, there is not only the  $SU(3)$  flavor symmetry of the quarks (which we have dealt with above) but also another  $SU(2)$  that originates from their spin. If we take both into account, quarks transform in the product irrep  $\mathbf{3} \otimes \mathbf{2}$  of  $SU(3) \times SU(2)$ . Explore this idea to obtain not only the pseudo-scalar meson octet from above but the baryon octet and decuplet, too.