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2. Sine-Gordon equation (17 points)

To be discussed on Tuesday, 10th October, 2023 in the tutorial.
Please indicate your preferences until Thursday, 05/10/2023, 21:00:00 on the website.

Exercise 2.1:

As our first introduction to Classical Field Theory, we start by studying the Sine-Gordon equation, which the describes the propagation of solitons (nonlinear but strongly stable localised wavepackets) and is also an integrable PDE (Partial Differential Equation).
Consider the Sine-Gordon equation

$$mR^2 \frac{\partial^2}{\partial t^2}\phi(x,t) = -mgR\sin\phi(x,t) + \tilde{\kappa}\frac{\partial^2}{\partial x^2}\phi(x,t), \quad x \in [x_{-M}, x_N]$$

with boundary conditions $\phi(x_{-M},t) = 0 = \phi'(x_{-M},t)$ and $\phi(x_N,t) = 2\pi n$, where M, N are fixed integers and n is an arbitrary integer.

a) (1 point) Using the following substitutions

$$\tau = \sqrt{\frac{g}{R}}t, \quad \xi = \sqrt{\frac{mgR}{\tilde{k}}}x, \quad \Phi(\xi,t) = \phi(x,t),$$

show that the Sine-Gordon equation can be written in the following standard form:

$$\frac{\partial^2}{\partial\tau^2}\Phi(\xi,\tau) - \frac{\partial^2}{\partial\xi^2}\Phi(\xi,\tau) + \sin\Phi(\xi,\tau) = 0, \qquad \Phi(\xi_{-M},\tau) = 0, \;\; \Phi(\xi_N,\tau) = 2\pi n. \tag{1}$$

b) (7 points) Find solutions of above equation assuming the ξ dependence only i.e. take $\Phi(\xi)$ (so called *static solutions*)

$$\Phi_\pm(\xi) = \pm 4\arctan(\exp(\xi - \xi_{-M})).$$

We are now introducing the Lorentz factor $\gamma = 1/\sqrt{1-v^2}$, $0 \leq |v| < 1$.

c) (4 points) Check that the function

$$\Phi_{+v}(\xi,\tau) = 4\arctan(\exp[\gamma(\xi - v\tau)])$$

is also a solution of (1).

d) (5 points) Do the same for the functions

$$\Phi_{++} = 4\arctan\left(\frac{v\sinh(\gamma\xi)}{\cosh(v\gamma\tau)}\right),$$
$$\Phi_{+-} = 4\arctan\left(\frac{\sinh(v\gamma\tau)}{v\cosh(\gamma\xi)}\right).$$