



1. Reminder of spin 0 and 1/2 fields (24 points)

To be discussed on Tuesday, 7th March, 2023 in the tutorial.
 Please indicate your preferences until Thursday, 02/03/2023, 21:00:00 on the website.

Exercise 1.1: Feynman Propagator of the Klein-Gordon Field

In the lecture, we stated that the propagator of the real scalar field has the form

$$D_F(x-y) = \lim_{\epsilon \rightarrow 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

We want to show that this propagator, as it should, matches the time-ordered correlation function.

- a) (3 points) Show that the Heaviside step function $\theta(x)$ has the integral representation

$$\theta(x) = \lim_{\epsilon \rightarrow 0^+} \mp \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\tau \frac{1}{\tau \pm i\epsilon} e^{\mp ix\tau}.$$

- b) (3 points) Using this representation, show

$$D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle.$$

Exercise 1.2: γ -matrices and Spinors

- a) (3 points) Use the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot 1$$

to show that the matrices

$$J^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$$

generate the Lie algebra $\mathfrak{so}(3,1)$ defined by the commutators

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}).$$

- b) (3 points) Show that the prescription for Dirac conjugation

$$\bar{\psi} = \psi^\dagger \gamma^0$$

renders the product $\bar{\psi}\psi$ a scalar under Lorentz transformation.

Exercise 1.3: Solutions of the Dirac Equation

In this exercise, we want to explore solutions of the Dirac equation

$$(i\gamma^\mu \partial_\mu - m) \psi = 0 \quad (1)$$

and its quantisation in more detail.

Hint: It might be a good idea to take a look at the sections 3.3 and 3.5 in Peskin and Schroeder while approaching this exercise.

a) (3 points) Use the plane wave ansatz

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4} [u(p)e^{-ipx} + v(p)e^{ipx}] , \quad p_0 > 0$$

to show that we can decompose (1) into the two parts

$$\begin{aligned} (\not{p} - m)u(p) &= 0 , \\ (-\not{p} - m)v(p) &= 0 . \end{aligned}$$

- b) (3 points) Each, $u(p)$ and $v(p)$, admits two linearly independent solutions, denoted as $u^s(p)$ and $v^s(p)$ with $s = 1, 2$. Compute them.
- c) (3 points) Finally, we want to understand the physical interpretation of these solution. Consider electrons and positrons, which are governed by the Dirac equation. How are they related to the solutions you derived above?
- d) (3 points) Calculate their spin at rest and show that it is directly related to the degeneracy which we labeled with s .