Quantum Field Theory, Summer 2024

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2. The Electromagnetic Field (18 points)

To be discussed on Friday, 8th March, 2024 in the tutorial.
Please indicate your preferences until Sunday, 03/03/2024, 21:00:00 on the website.

Exercise 2.1: Classical Electrodynamics with Sources

Classical electrodynamics without sources is governed by the action

$$S_1 = -\frac{1}{4}\int \mathrm{d}^4x F_{\mu\nu}F^{\mu\nu}\,, \qquad \text{with} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{1}$$

In the lecture, we have seen that it can be coupled to a complex scalar field by introducing the additional interaction term

$$S_2 = e\int \mathrm{d}^4x A_\mu J^\mu\,,$$

where J^μ is a conserved current. To show how this formulation gives rise to the expected dynamics of the electromagnetic field:

a) (3 points) Derive the homogeneous Maxwell equations from the least action principle for the action S_1. Treat the components of $A_\mu(x)$ as the dynamical variables. Write the equations in standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$.

b) (3 points) Now combine the two actions S_1 and S_2 and show that their field equations result in the inhomogeneous Maxwell equations. What is the physical interpretation of J^μ's components? We learned that J^μ is a conserved current governed by

$$\partial_\mu J^\mu = 0\,,$$

and that it gives rise to the conserved charge

$$Q = \int \mathrm{d}^3x J^0.$$

What is the interpretation of these two equations in classical electrodynamics?

Exercise 2.2: Quantisation of the Electromagnetic Field

In the lecture we had a tight schedule and therefore glossed quickly over the quantisation of the electromagnetic field. Here, we want to take a closer look at some important details. *Note: We use the gauge fixing*

$$A_0 = 0 \qquad \textit{and} \qquad \partial_i A^i = 0\,. \tag{2}$$

a) (3 points) Starting from the action (1), compute the canonical momentum Π^μ and the Hamiltonian H.

b) (3 points) In the only non-vanishing commutator,

$$[A^i(\vec{x}), \Pi^j(\vec{y})] = i \int \frac{\mathrm{d}^2 p}{(2\pi)^3} \left(\delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2} \right) e^{i\vec{p}(\vec{x}-\vec{y})}$$

the second term looks peculiar. Show that it is required in order to not break the gauge condition $\partial_i A^i = 0$.

c) (3 points) Use the mode expansion

$$\vec{A}(\vec{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \, 2E_0} \sum_{\lambda=1}^{2} \vec{\epsilon}^{\lambda}(p) \left[a_p^{\lambda} e^{-ip_\mu x^\mu} + (a_p^{\lambda})^\dagger e^{-ip_\mu x^\nu} e^{ip_\mu x^\mu} \right]$$

to show that the creation and annihilation operators $(a_p^{\lambda})^\dagger$ and a_p^{λ} just describe uncoupled harmonic oscillators. Derive the constraints on the polarisation vector $\vec{\epsilon}^{\lambda}$ imposed by the gauge fixing $\partial_i A^i = 0$.

Exercise 2.3: Coupling of Spin 1/2 Particles to the Electromagnetic Field 3 points

Show that the action

$$S = \int \mathrm{d}^4 x \overline{\Psi} (i\partial\!\!\!/ - m) \Psi \tag{3}$$

has a global U(1) symmetry acting by

$$\Psi \to e^{i\Lambda} \Psi \,. \tag{4}$$

Compute the conserved current J^μ for this symmetry and interpret it using the results from b.