Selected Tools of Modern Theoretical Physics 2B, Summer 2025

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2. Mathematical foundations (17 points)

To be discussed on Wednesday, 30^{th} April, 2025 in the tutorial.
Please indicate your preferences until Friday, 25/04/2025, 21:00:00 on the website.

Exercise 2.1: Groups

a) (2 points) Determine if the following sets with the specified operation form a group. If not, state at least one group axiom that fails.

- the set of integers $\mathbb{Z}$ under subtraction $-$.
- the set of rational numbers $\mathbb{Q} \setminus \{0\}$ under multiplication $\times$.
- the set of 2×2 real matrices with determinant 1 (i.e. $SL(2, \mathbb{R})$) under matrix multiplication.
- the set of integers $\mathbb{Z}$ under the operation $a \star b = a + b + 1$.

b) (1 point) Prove that for any group $(G, *)$, the identity element e is unique.

Exercise 2.2: Fields

a) (2 points) Explain why the set of integers $(\mathbb{Z}, +, \times)$ with the usual addition and multiplication is *not* a field. Which axiom(s) fail?

b) (2 points) Consider the set $A = \{0, 1\}$ with addition and multiplication defined modulo 2. Verify that $(A, +, \times)$ is a field.

Exercise 2.3: Vector spaces

a) (2 points) Determine if the following sets are vector spaces over the specified field. If not, explain why.

- $V = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}$ over the field $\mathbb{R}$, with standard vector addition and scalar multiplication.
- $V = \{f : \mathbb{R} \to \mathbb{R} | f(0) = 1\}$ over the field $\mathbb{R}$, with standard function addition and scalar multiplication.
- The set of 2×2 real matrices $M_2(\mathbb{R})$ over the field $\mathbb{R}$ with standard matrix addition and scalar multiplication.
- The set $SO(3)$ over the field $\mathbb{R}$ with standard matrix addition and scalar multiplication.

b) (2 points) Let $P_2(\mathbb{R})$ be the set of polynomials with real coefficients of degree at most 2, i.e. $P_2(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 | a_0, a_1, a_2 \in \mathbb{R}\}$. Show that $P_2(\mathbb{R})$ is a vector space over $\mathbb{R}$ with standard polynomial addition and scalar multiplication. What is its dimension? Find a basis.

Exercise 2.4: Combining vector spaces

Let $V = \mathbb{R}^4$, and define the subspaces $U = \text{span}\{(1,0,1,0),(0,1,0,1)\}$, $W = \text{span}\{(1,0,-1,0),(0,1,0,-1)\}$

a) (1 point) Show that $U \cap W = \{0\}$.

b) (1 point) Show that $U + W = V$, so along with the previous property we have $U \oplus W = V$.

Let $X = \mathbb{R}^2$, with the standard unit basis $\{e_1, e_2\}$. Consider $X \otimes X$.

a) (1 point) Decompose $X \otimes X$ into symmetric and antisymmetric parts.

b) (1 point) What is the dimension of each part?

c) (2 points) Generalise these results to $X = R^n$.