



2. The Electromagnetic Field (18 points)

To be discussed on Monday, 14th March, 2022 in the tutorial.

Please indicate your preferences until Wednesday, 09/03/2022, 21:00:00 on the website.

Exercise 2.1: Classical Electrodynamics with Sources

Classical electrodynamics without sources is governed by the action

$$S_1 = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu}, \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (1)$$

In the lecture, we have seen that it can be coupled to a complex scalar field by introducing the additional interaction term

$$S_2 = e \int d^4x A_\mu J^\mu,$$

where J^μ is a conserved current. To show how this formulation gives rise to the expected dynamics of the electromagnetic field:

- a) (3 points) Derive the homogeneous Maxwell equations from the least action principle for the action S_1 . Treat the components of $A_\mu(x)$ as the dynamical variables. Write the equations in standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.
- b) (3 points) Now combine the two actions S_1 and S_2 and show that their field equations result in the inhomogeneous Maxwell equations. What is the physical interpretation of J^μ 's components? We learned that J^μ is a conserved current governed by

$$\partial_\mu J^\mu = 0.$$

It gives rise to the conserved charge

$$Q = \int d^3x J^0.$$

What is the interpretation of these two equations in classical electrodynamics.

Exercise 2.2: Quantisation of the Electromagnetic Field

In the lecture we had a tight schedule and therefore glossed quickly over the quantisation of the electromagnetic field. Here, we want to take a closer look at some important details. *Note: We use the gauge fixing*

$$A_0 = 0 \quad \text{and} \quad \partial_i A^i = 0. \quad (2)$$

- a) (3 points) Starting from the action (1), compute the canonical momentum Π^μ and the Hamiltonian H .

b) (3 points) In the only non-vanishing commutator,

$$[A^i(\vec{x}), \Pi^j(\vec{y})] = i \int \frac{d^2p}{(2\pi)^3} \left(\delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2} \right) e^{i\vec{p}(\vec{x}-\vec{y})}$$

the second term looks peculiar. Show that it is required to not break that gauge condition $\nabla_i A^i = 0$.

c) (3 points) Use the mode expansion

$$\vec{A}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3 2E_0} \sum_{\lambda=1}^2 \vec{\epsilon}^\lambda(p) [a_p^\lambda e^{-ip_\mu x^\mu} + (a_p^\lambda)^\dagger e^{-ip_\mu x^\nu} e^{ip_\mu x^\mu}]$$

to show that the creation and annihilation operators $(a_p^\lambda)^\dagger$ and a_p^λ just describe uncoupled harmonic oscillators. Derive the constraints on the polarisation vector $\vec{\epsilon}^\lambda$ imposed by the gauge fixing $\partial_i A^i$.

Exercise 2.3: Coupling of Spin 1/2 Particles to the Electromagnetic Field 3 points

Show that the action

$$S = \int d^4x \bar{\Psi} (i\not{\partial} - m) \Psi \quad (3)$$

has a global U(1) symmetry acting by

$$\Psi \rightarrow e^{i\Lambda} \Psi. \quad (4)$$

Compute the conserved current J^μ for this symmetry and interpret it using the results from b.