Quantum Field Theory, Summer 2024

Lecturer: Dr. Falk Hassler, falk.hassler@uwr.edu.pl
Tutorials: Prof. UWr Pok Man Lo, pokman.lo@uwr.edu.pl
Assistant: M.Sc. Alex Swash, alex.swash@uwr.edu.pl

1. Reminder of spin 0 and 1/2 fields (24 points)

To be discussed on Friday, 1st March, 2024 in the tutorial.
Please indicate your preferences until Sunday, 25/02/2024, 21:00:00 on the website.

Exercise 1.1: Feynman Propagator of the Klein-Gordon Field

In the lecture, we stated that the propagator of the real scalar field has the form

$$D_{\mathrm{F}}(x-y) = \lim_{\epsilon\to 0^+} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

We want to show that this propagator, as it should, matches the time-ordered correlation function.

a) (3 points) Show that the Heaviside step function $\theta(x)$ has the integral representation

$$\theta(x) = \lim_{\epsilon\to 0^+} \mp\frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathrm{d}\tau \, \frac{1}{\tau \pm i\epsilon} e^{\mp i x\tau} \, .$$

b) (3 points) Using this representation, show

$$D_{\mathrm{F}}(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle \, .$$

Exercise 1.2: γ-matrices and Spinors

a) (3 points) Use the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot 1$$

to show that the matrices

$$J^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

generate the Lie algebra $\mathfrak{so}(3,1)$ defined by the commutators

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right) \, .$$

b) (3 points) Show that the prescription for Dirac conjugation

$$\overline{\psi} = \psi^\dagger \gamma^0$$

renders the product $\overline{\psi}\psi$ a scalar under Lorentz transformation.

Exercise 1.3: Solutions of the Dirac Equation

In this exercise, we want to explore solutions of the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0 \tag{1}$$

and its quantisation in more detail.

Hint: It might be a good idea to take a look at the sections 3.3 and 3.5 in Peskin and Schroeder while approaching this exercise.

a) (3 points) Use the plane wave ansatz

$$\psi(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left[u(p)e^{-ipx} + v(p)e^{ipx}\right]\,, \quad p_0 > 0$$

to show that we can decompose (1) into the two parts

$$\begin{aligned}(\not{p} - m)u(p) &= 0\,,\\ (-\not{p} - m)v(p) &= 0\,.\end{aligned}$$

b) (3 points) Each, $u(p)$ and $v(p)$, admits two linearly independent solutions, denoted as $u^s(p)$ and $v^s(p)$ with $s = 1, 2$. Compute them.

c) (3 points) Finally, we want to understand the physical interpretation of these solutions. Consider electrons and positrons, which are governed by the Dirac equation. How are they related to the solutions you derived above?

d) (3 points) Calculate their spin at rest and show that it is directly related to the degeneracy which we labeled with s.