Selected Tools of Modern Theoretical Physics 2B, Summer 2025

Lecturer: Dr. Falk Hassler, falk.hassler@uwr.edu.pl
Tutorials: M.Sc. Alex Swash, alex.swash@uwr.edu.pl

1. Matrix groups (15 points)

To be discussed on Monday, 28th April, 2025 in the tutorial.
Please indicate your preferences until Wednesday, 23/04/2025, 21:00:00 on the website.

Exercise 1.1: SO(2) and U(1)

Consider the group SO(2), i.e. the group of 2×2 real matrices satisfying the following two conditions,

$$A^T A = I_{2\times 2}, \qquad \det A = 1. \tag{1}$$

Show that

a) (1 point) each such matrix can be written in the form

$$A = A(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \quad \alpha \in \mathbb{R} \tag{2}$$

b) (1 point) the group SO(2) is isomorphic to the circle S^1, i.e. find a map $f : \mathbb{R} \to \mathbb{R}^2$ realising this isomorphism.

c) (1 point) Consider now the group U(1), i.e. the group of 1×1 complex matrices (so complex numbers), satisfying the condition $A^\dagger A = 1$.
Show that each such "matrix" can be written in the form

$$A = A(\alpha) = e^{i\alpha} = \cos\alpha + i\sin\alpha, \qquad \alpha \in \mathbb{R}. \tag{3}$$

d) (1 point) Show that the group U(1) is also isomorphic to the circle S^1.

e) (1 point) Show that the representation (2) is in fact reducible, by diagonalising the matrix $A(\alpha)$. What are the irreducible representations you obtain?

Exercise 1.2: su(2) and so(3)

SU(2) is the group of 2×2 complex matrices which are unitary and have unit determinant, i.e. satisfying the conditions $A^\dagger A = I_{2\times 2}$ and $\det A = 1$.

a) (1 point) By expanding $A \simeq I + X$ and leading order terms in X, what conditions must the matrix X satisfy for A to lie in the group SU(2)?

b) (1 point) Such matrices X now form the Lie algebra $\mathfrak{su}(2)$. Show that any matrix in $\mathfrak{su}(2)$ is of the form:

$$X = i(a\sigma_1 + b\sigma_2 + c\sigma_3), \tag{4}$$

where $a, b, c \in \mathbb{R}$ and the σ_i are the Pauli matrices.

c) (2 points) Compute the commutators $[\sigma_i, \sigma_j]$ and argue why $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ (which we discussed in the lecture) are isomorphic as Lie algebras.

Exercise 1.3: SO(4) and so(4)

Consider now SO(4), the group of 4×4 real orthogonal matrices of unit determinant.

a) (2 points) How many real parameters (degrees of freedom) does SO(4) have? Can you generalise this result to SO(n)?

b) (1 point) Let $A \simeq I + X$ be a matrix in SO(4) expanded around the identity. What conditions must the matrix X satisfy for A to lie in SO(4)?

c) (2 points) The set of such matrices X forms the Lie algebra $\mathfrak{so}(4)$. What is its dimension? What about $\mathfrak{so}(n)$? *Hint: Count the degrees of freedom of* X*.*

d) (1 point) It turns out that there is an isomorphism (which is very important for theoretical physics)

$$\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2). \tag{5}$$

Try to count dimensions on both sides. Do they match?