



## 1. Matrix groups (15 points)

To be discussed on Monday, 11<sup>th</sup> May, 2026 in the tutorial.

Please indicate your preferences until Wednesday, 06/05/2026, 21:00:00 on the website.

### Exercise 1.1: SO(2) and U(1)

Consider the group SO(2), i.e. the group of  $2 \times 2$  real matrices satisfying the following two conditions,

$$A^T A = I_{2 \times 2}, \quad \det A = 1. \quad (1)$$

Show that

- a) (1 point) each such matrix can be written in the form

$$A = A(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}, \quad \alpha \in \mathbb{R} \quad (2)$$

- b) (1 point) the group SO(2) is isomorphic to the circle  $S^1$ , i.e. find a map  $f : \mathbb{R} \rightarrow \mathbb{R}^2$  realising this isomorphism.
- c) (1 point) Consider now the group U(1), i.e. the group of  $1 \times 1$  complex matrices (so complex numbers), satisfying the condition  $A^\dagger A = 1$ .

Show that each such "matrix" can be written in the form

$$A = A(\alpha) = e^{i\alpha} = \cos \alpha + i \sin \alpha, \quad \alpha \in \mathbb{R}. \quad (3)$$

- d) (1 point) Show that the group U(1) is also isomorphic to the circle  $S^1$ .
- e) (1 point) Show that the representation (2) is in fact reducible, by diagonalising the matrix  $A(\alpha)$ . What are the irreducible representations you obtain?

### Exercise 1.2: su(2) and so(3)

SU(2) is the group of  $2 \times 2$  complex matrices which are unitary and have unit determinant, i.e. satisfying the conditions  $A^\dagger A = I_{2 \times 2}$  and  $\det A = 1$ .

- a) (1 point) By expanding  $A \simeq I + X$  and leading order terms in  $X$ , what conditions must the matrix  $X$  satisfy for  $A$  to lie in the group SU(2)?
- b) (1 point) Such matrices  $X$  now form the Lie algebra  $\mathfrak{su}(2)$ . Show that any matrix in  $\mathfrak{su}(2)$  is of the form:

$$X = i(a\sigma_1 + b\sigma_2 + c\sigma_3), \quad (4)$$

where  $a, b, c \in \mathbb{R}$  and the  $\sigma_i$  are the Pauli matrices.

- c) (2 points) Compute the commutators  $[\sigma_i, \sigma_j]$  and argue why  $\mathfrak{su}(2)$  and  $\mathfrak{so}(3)$  (which we discussed in the lecture) are isomorphic as Lie algebras.

### Exercise 1.3: $\mathrm{SO}(4)$ and $\mathfrak{so}(4)$

Consider now  $\mathrm{SO}(4)$ , the group of  $4 \times 4$  real orthogonal matrices of unit determinant.

- a) (2 points) How many real parameters (degrees of freedom) does  $\mathrm{SO}(4)$  have? Can you generalise this result to  $\mathrm{SO}(n)$ ?
- b) (1 point) Let  $A \simeq I + X$  be a matrix in  $\mathrm{SO}(4)$  expanded around the identity. What conditions must the matrix  $X$  satisfy for  $A$  to lie in  $\mathrm{SO}(4)$ ?
- c) (2 points) The set of such matrices  $X$  forms the Lie algebra  $\mathfrak{so}(4)$ . What is its dimension? What about  $\mathfrak{so}(n)$ ? *Hint: Count the degrees of freedom of  $X$ .*
- d) (1 point) It turns out that there is an isomorphism (which is very important for theoretical physics)

$$\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2). \quad (5)$$

Try to count dimensions on both sides. Do they match?