Classical Field Theory, Winter 2025/26

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1. Field theory in two dimensions (20 points)

To be discussed on Wednesday, 15th October, 2025 in the tutorial.
Please indicate your preferences until Friday, 10/10/2025, 21:00:00 on the website.

Exercise 1.1: Sine-Gordon equation

As our first introduction to Classical Field Theory, we start by studying the Sine-Gordon equation, which the describes the propagation of solitons (nonlinear but strongly stable localised wavepackets) and is also an integrable PDE (Partial Differential Equation).
Consider the Sine-Gordon equation

$$mR^2 \frac{\partial^2}{\partial t^2}\phi(x,t) = -mgR\sin\phi(x,t) + \tilde{\kappa}\frac{\partial^2}{\partial x^2}\phi(x,t), \qquad x \in [x_{-M}, x_N]$$

with boundary conditions $\phi(x_{-M},t) = 0$ and $\phi(x_N,t) = 2\pi n$, where M, N are fixed integers and n is an arbitrary integer.

a) (1 point) Using the following substitutions

$$\tau = \sqrt{\frac{g}{R}}t, \qquad \xi = \sqrt{\frac{mgR}{\tilde{k}}}x, \qquad \Phi(\xi,t) = \phi(x,t),$$

show that the Sine-Gordon equation can be written in the following standard form:

$$\frac{\partial^2}{\partial \tau^2}\Phi(\xi,\tau) - \frac{\partial^2}{\partial \xi^2}\Phi(\xi,\tau) + \sin\Phi(\xi,\tau) = 0, \qquad \Phi(\xi_{-M},\tau) = 0, \;\; \Phi(\xi_N,\tau) = 2\pi n. \tag{1}$$

We are now going to find solutions of the above equation assuming ξ dependence only i.e. take $\Phi(\xi)$ (so called *static solutions*)

b) (3 points) Show that (1) reduces to

$$\frac{d\Phi}{d\xi} = \pm 2\sin\frac{\Phi}{2}\,, \tag{2}$$

with the initial conditions

$$\left.\frac{d\Phi}{d\xi}\right|_{\xi=\xi_{-M}} = \pm \left.\frac{d\Phi}{d\xi}\right|_{\xi=\xi_N}$$

c) (3 points) Pushing (2) a bit further, show that it can be written in the following form:

$$\ln\left(\tan\frac{\Phi}{4}\right) = \pm(\xi - \xi_0)\,, \tag{3}$$

where ξ_0 is an integration constant.

d) (2 points) Making use of the identity

$$\arctan\frac{1}{x} = \frac{\pi}{2} - \arctan x\,,$$

show that (3) finally reduces to

$$\Phi(\xi) = \pm 4\arctan(\exp(\xi - \xi_0))\,. \tag{4}$$

e) (2 points) Does the form of (4) violate the boundary conditions, or any of the previous results? If so, what assumptions have to be made for ξ_{-M} and ξ_N? Can you also make an assume the value of n in (1)?

We are now introducing the Lorentz factor $\gamma = 1/\sqrt{1-v^2}$, $0 \leq |v| < 1$.

f) (4 points) Check that the function

$$\Phi_{+v}(\xi,\tau) = 4\arctan(\exp[\gamma(\xi - v\tau)])$$

is also a solution of (1).

g) (5 points) Do the same for the functions

$$\begin{aligned}
\Phi_{++} &= 4\arctan\left(\frac{v\sinh(\gamma\xi)}{\cosh(v\gamma\tau)}\right),\\
\Phi_{+-} &= 4\arctan\left(\frac{\sinh(v\gamma\tau)}{v\cosh(\gamma\xi)}\right).
\end{aligned}$$