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# 2. Abelian and non-Abelian gauge theory (18 points)

To be discussed on Friday, 11th October, 2024 in the tutorial.
Please indicate your preferences until Sunday, 06/10/2024, 21:00:00 on the website.

### Exercise 2.1: Global symmetries

Consider the action for a U(1) gauge theory

$$S = -\frac{1}{4g^2}\int \mathrm{d}^4x F_{\mu\nu}F^{\mu\nu} \qquad \text{with} \qquad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}\,. \tag{1}$$

Under a Poincaré transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$, the gauge field $A_\mu$ transforms as

$$A^\mu(x) \to A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x - a)\,. \tag{2}$$

a) (2 points) Show that the action (1) is invariant under (2). Moreover, derive the infinitesimal transformation law

$$\delta A_\mu = \omega_{\nu\lambda}x^\nu\partial^\lambda A_\mu - \omega_\mu{}^\nu A_\nu - a^\nu\partial_\nu A_\mu$$

by using $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$ with $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

b) (3 points) Derive the conserved current, $\Theta_{\mu\nu}$ satisfying $\partial^\mu\Theta_{\mu\nu} = 0$, corresponding to infinitesimal translations with the parameter $a^\mu$. You will note that it is not symmetric. However, we can construct a symmetric tensors which is still conserved, namely

$$T_{\mu\nu} = \Theta_{\mu\nu} + \partial^\lambda f_{\lambda\mu\nu}\,,$$

by choosing $f_{\mu\nu\lambda} = -f_{\nu\mu\lambda}$ appropriately. Show that $f_{\lambda\mu\nu} = -\frac{1}{g^2}A_\nu F_{\mu\lambda}$ works and will give the canonical energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{g^2}\left(F_{\mu\lambda}F_\nu{}^\lambda - \tfrac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

of electrodynamics.

c) (2 points) Derive the conserved current for Lorentz transformations with the parameter $\omega_{\mu\nu}$. It is of the form

$$N^{\nu\kappa\lambda} = x^\kappa T^{\lambda\nu} - x^\lambda T^{\kappa\nu}\,.$$

Check the this current is conserved and therefore satisfies $\partial^\mu N_{\mu\nu\lambda} = 0$ for any solutions of the field equations.

### Exercise 2.2: Yang-Mills theory

Now, we look at the action of a non-Abelian gauge theory

$$S = -\frac{1}{2g^2}\int \mathrm{d}^4x\ \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \tag{3}$$

with the field strength $F_{\mu\nu} = F^a_{\mu\nu}T_a = 2\partial_{[\mu}A_{\nu]} + i[A_\mu, A_\nu]$ and the gauge potential $A_\mu = A^a_\mu T_a$.

a) (2 points) Assuming real structure coefficient ${f_{ab}}^c$ in the commutator

$$[T_a, T_b] = i{f_{ab}}^c T_c\,,$$

show that the generators of the underlying Lie algebra $T_a$ can be chosen to be hermitian. Is it possible to generalize this choice? If yes, how? Argue why the pairing

$$Tr(T_a T_b) = \kappa_{ab}$$

is real, too.

b) (2 points) Show that the field strength transforms as

$$F_{\mu\nu} \to F'_{\mu\nu} = U F_{\mu\nu} U^\dagger$$

assuming the transformation

$$A_\mu \to A'_\mu = U A_\mu U^\dagger - iU\partial_\mu U^\dagger$$

with $U = \exp\left[i\alpha^a(x)T_a\right]$ and real, coordinate dependent parameters $\alpha^a(x)$.

c) (2 points) Show that the equations of motion for the action (3) can be written as

$$D_\mu F^{\mu\nu} = 0 \qquad \text{with} \qquad D_\mu F^{\nu\rho} = \partial_\mu F^{\nu\rho} + i[A_\mu, F^{\nu\rho}]\,.$$

d) (2 points) Verify that the Bianchi identity

$$3D_{[\mu}F_{\nu\rho]} = 0$$

automatically holds.

e) (3 points) As we consider here four dimensions, a term of the form

$$S^{\text{top}} = \int \mathrm{d}^4x \frac{\vartheta}{16\pi^2}\,\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \tag{4}$$

may also be added to the action, with $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ denoting the dual field strength tensor (dual under Hodge duality). Here, we encounter the totally antisymmetric tensors $\epsilon^{\mu\nu\rho\sigma}$ in four dimensions. It is normalized such that $\epsilon^{0123} = -1$. Show that (4) is also gauge invariant but does not contribute to the equations of motion.