Classical Field Theory, Winter 2023/24

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13. Lagrangian densities (23 points)

To be discussed on Wednesday, 17^{th} January, 2024 in the tutorial.
Please indicate your preferences until Friday, 12/01/2024, 21:00:00 on the website.

Exercise 13.1: Equations of motion

Find the Euler-Lagrange equations for the following Lagrangian densities:

a) (2 points)

$$\mathcal{L} = -(\partial_\mu A^\nu)(\partial_\nu A^\mu) + \frac{1}{2}m^2 A_\mu A^\mu + \frac{\lambda}{2}(\partial_\mu A^\mu)^2.$$

b) (2 points)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}A_\mu A^\mu,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

c) (2 points)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\phi^2 - \frac{\lambda}{4}\phi^4,$$

where $\phi(x)$ is a real scalar field.

d) (2 points)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu \phi - ieA_\mu \phi)(\partial^\mu \phi^* + ieA^\mu \phi^*) - \frac{1}{2}m^2 \phi^* \phi,$$

where $\phi(x)$ is a complex scalar field.

Exercise 13.2: Equation of motion invariance

a) (4 points) Show that when the divergence of an arbitrary field function $f^\mu(\phi_i)$ is added to a Lagrangian density, its equations of motion remain unchanged.
Hint: Assume that the variation commutes with the partial derivatives and the variation of the fields ϕ_i vanish at infinity.

Exercise 13.3: Equation of motion invariance

The Lagrangian density of a real three component scalar field is given in the following form

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^T \partial^\mu \phi - \frac{1}{2}m^2 \phi^T \phi, \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix},$$

where T denotes transposition.

a) (2 points) Find the equations of motion.

b) (3 points) Check if $\mathcal{L}$ is invariant under $SO(3)$.

Hint: Assume rigid solutions, that is for R being an $SO(3)$ rotation matrix, $\partial_\mu R = 0$ holds. The rotation matrices are given by

$$R_1(\alpha_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & -\sin\alpha_1 \\ 0 & \sin\alpha_1 & \cos\alpha_1 \end{pmatrix} = e^{\alpha_1 T_1},$$

$$R_2(\alpha_2) = \begin{pmatrix} \cos\alpha_2 & 0 & \sin\alpha_2 \\ 0 & 1 & 0 \\ -\sin\alpha_2 & 0 & \cos\alpha_2 \end{pmatrix} = e^{\alpha_2 T_2},$$

$$R_3(\alpha_3) = \begin{pmatrix} \cos\alpha_3 & -\sin\alpha_3 & 0 \\ \sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{\alpha_3 T_3}.$$

We are now going to find the Noether currents, which are given by

$$J_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_b)} \left(\frac{\delta \phi_b}{\delta \omega^a} - \frac{\delta x^\mu}{\delta \omega^a} \partial_\mu \phi_b \right) + \mathcal{L} \frac{\delta x^\mu}{\delta \omega^a}, \qquad a = 1, 2, 3.$$

c) (3 points) Prove

$$\delta\phi_a = \phi_a' - \phi_a = \epsilon_{abc} \alpha^b \phi^c.$$

Hint: Show that the transformed under $SO(3)$ fields for small α_i can be written as

$$\phi_a' \simeq \phi_a + \delta\omega_i \cdot (T_i)_{ba} \phi^b,$$

where

$$R_i(\omega_i) = e^{\delta\omega_i T_i}, \qquad T_i = \left. \frac{\partial R_i}{\partial \alpha_i} \right|_{\alpha_i = 0}.$$

d) (3 points) Show that the Noether currents can be compactly written as

$$J_a^\mu = \epsilon_{abc} \phi^b \partial^\mu \phi^c.$$