Classical Field Theory, Winter 2023/24

Lecture: Prof. Andrzej Frydryszak, andrzej.frydryszak@uwr.edu.pl
Tutorial: Dr. Falk Hassler, falk.hassler@uwr.edu.pl
Assistant: M.Sc. Achilles Gitsis, achilleas.gitsis@uwr.edu.pl

12. Electrodynamics (20 points)

To be discussed on Wednesday, 10th January, 2024 in the tutorial.
Please indicate your preferences until Friday, 05/01/2024, 21:00:00 on the website.

Exercise 12.1: Covariant electrodynamics

Let

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and explicit identifications of the electric and magnetic fields are given by

$$F^{0i} = -E^i$$
$$F^{ij} = -\epsilon^{ijk}B_k.$$

a) (3 points) Prove that the equations of motion are given by

$$\partial_\mu F^{\mu\nu} = 0.$$

b) (3 points) Prove that the canonical energy-momentum tensor for the Lagrangian density defined above is given by

$$T^{\mu\nu} = \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^{\mu\lambda}F^{\nu}{}_{\lambda} - F^{\mu\lambda}\partial_\lambda A^\nu.$$

c) (2 points) Show that $T^{\mu\nu}$ is not symmetric.

d) (3 points) One can remedy this by considering a new tensor

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}, \qquad K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu.$$

Find out that $\tilde{T}^{\mu\nu}$ is symmetric.

e) (3 points) Show that from $\tilde{T}^{\mu\nu}$ one can reproduce the standard expression for the energy density, that is

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2).$$

Hint: Use $\epsilon_{ijk}\epsilon^{ijl} = 2\delta_k^l$.

f) (3 points) Do the same as in the previous task but with the momentum density, i.e.

$$\vec{S} = \vec{E} \times \vec{B}.$$

g) (3 points) Calculate the trace of $\tilde{T}^{\mu\nu}$.