Classical Field Theory, Winter 2023/24

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11. Pauli-Lubański vector and relativistic particle action (16 points)

To be discussed on Wednesday, 3rd January, 2024 in the tutorial.
Please indicate your preferences until Friday, 29/12/2023, 21:00:00 on the website.

Exercise 11.1: Pauli-Lubański vector

The Pauli-Lubański vector is defined as

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma .$$

Show that:

a) (3 points)

$$W_\mu P^\mu = 0 .$$

Hint: Keep in mind that $\epsilon_{\mu\nu\rho\sigma}$ is antisymmetric under the exchange of μ and σ, which means that for a generic tensor $T^{\mu\sigma}$ we can write

$$\epsilon_{\mu\nu\rho\sigma} T^{\mu\sigma} = \frac{1}{2}(\epsilon_{\mu\nu\rho\sigma} - \epsilon_{\sigma\nu\rho\mu}) T^{\mu\sigma} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(T^{\mu\sigma} - T^{\sigma\mu}) .$$

b) (3 points)

$$[W_\mu, P_\alpha] = 0 .$$

Hint: $\epsilon_{\mu\nu\rho\sigma}$ is antisymmetric under the exchange of ν and σ as well so you can use the same trick as in the previous task. Also,

$$[AB, C] = A[B, C] + [A, C]B .$$

c) (3 points)

$$W^2 = -\frac{1}{2} M_{\mu\nu} M^{\mu\nu} P_\sigma P^\sigma + M_{\mu\sigma} M^{\nu\sigma} P^\mu P_\nu .$$

Hint: Use that

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\alpha\beta\gamma} = -\delta^\alpha_\nu\delta^\beta_\rho\delta^\gamma_\sigma + \delta^\alpha_\rho\delta^\beta_\nu\delta^\gamma_\sigma - \delta^\alpha_\rho\delta^\beta_\sigma\delta^\gamma_\nu + \delta^\alpha_\sigma\delta^\beta_\rho\delta^\gamma_\nu - \delta^\alpha_\sigma\delta^\beta_\nu\delta^\gamma_\rho + \delta^\alpha_\nu\delta^\beta_\sigma\delta^\gamma_\rho .$$

d) (3 points) P^2 and W^2 commute with every generator of the Poincaré group.

Exercise 11.2: Relativistic particle action

Consider the following action for a relativistic particle with charge q_0 in an external field A_μ

$$S = -\frac{1}{m}\int \left(\frac{p_\mu}{2} + \frac{q_0}{c}A_\mu(x)\right) p^\mu d\tau, \qquad p^\mu = m\frac{dx^\mu}{d\tau} .$$

a) (4 points) Show that the condition $\delta S = 0$ yields the following equations of motion:

$$\frac{dp_\mu}{d\tau} = \frac{q_0}{c} F_{\mu\nu} \frac{dx^\nu}{d\tau}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Hint: Don't forget that

$$\delta \frac{dx^\mu}{d\tau} = \frac{d}{d\tau} \delta x^\mu, \qquad \delta(A_\mu(x)) = \partial_\nu A_\mu \delta x^\nu.$$