Classical Field Theory, Winter 2023/24

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10. Poincaré group (20 points)

To be discussed on Wednesday, 20th December, 2023 in the tutorial.
Please indicate your preferences until Friday, 15/12/2023, 21:00:00 on the website.

Exercise 10.1: Unitary representation of the Poincaré group

Let $U(\Lambda, a)$ be a unitary representation of the Poincaré group with the multiplication law

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1),$$

where $\Lambda \in \mathcal{L}_+^\uparrow$, $\Lambda = (\Lambda^\mu{}_\nu)$ and $a = (a^\mu)$. Moreover, let

$$U_\rho(\omega, \epsilon) = e^{\frac{1}{2}\omega^{\mu\nu}M_{\mu\nu} + \epsilon^\mu P_\mu},$$

such that

$$U(\Lambda(\omega), 0) = U_\rho(\omega, 0)$$
$$U(\mathbb{1}, a(\epsilon)) = U_\rho(0, \epsilon),$$

because

$$\Lambda \simeq \mathbb{1} + \omega + \mathcal{O}(\omega^2)$$

and also $\omega^{\mu\nu} = -\omega^{\nu\mu}$, $M_{\mu\nu} = -M_{\nu\mu}$.

a) (3 points) Show that

$$U^{-1}(\Lambda, 0)U(\mathbb{1}, \epsilon)U(\Lambda, 0) = U(\mathbb{1}, \Lambda^{-1}\epsilon).$$

Hint: Don't forget that the inverse $U^{-1}(\Lambda, a)$ is also an element of the Poincaré group that shall satisfy

$$U^{-1}(\Lambda, a)U(\Lambda, a) = U(\Lambda, a)U^{-1}(\Lambda, a) = U(\mathbb{1}, 0).$$

b) (3 points) Show that

$$U^{-1}(\Lambda, 0)P_\mu U(\Lambda, 0) = (\Lambda^{-1})^\nu{}_\mu P_\nu.$$

Hint: Try to relate this with the previous task and U_ρ.

c) (4 points) Find the commutator $[M_{\mu\nu}, P_\rho]$.

Hint: Expand the relation of the previous task in 1st order in ω and keep terms up to ω. Also, keep in mind that a generic tensor $T_{\mu\nu}$ can be written in terms of a symmetric and an antisymmetric component, that is

$$T_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) + \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}).$$

d) (3 points) Show that

$$U^{-1}(\Lambda, 0)U(\Lambda', 0)U(\Lambda, 0) = U(\Lambda^{-1}\Lambda'\Lambda, 0).$$

Hint: Notice the similarity with task a).

e) (4 points) Find the commutator $[M_{\mu\nu}, M_{\rho\sigma}]$.

Hint: Expand the relation of the previous task in 1st order in ω, ω' *and keep terms up to* ω, ω' *and* $\omega\omega'$.

f) (3 points) Show that $[P_\mu, P_\nu] = 0$.

Hint: Try to do so via calculating $[U(\mathbb{1}, a), U(\mathbb{1}, a')]$.