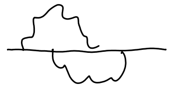
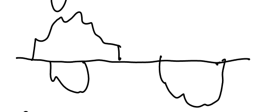
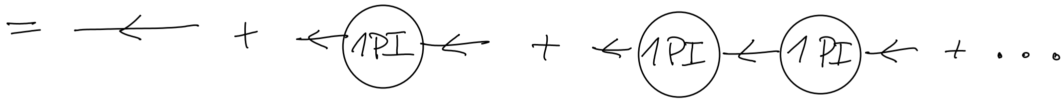
Def.: A one-particle irreducible (1PI) diagram cannot be split in two by removing a single line.

Example:  is 1PI,  is not.

$-i\Sigma(p)$ is the sum of all 1PI diagrams with two external lines.

$-i\Sigma(p) =$ 

We can now write $\int d^4x \, \langle 0 | T \psi(x) \bar{\psi}(0) | 0 \rangle e^{ipx}$



$$= \frac{i}{\not{p} - m_0 - \Sigma(\not{p})} \quad \text{see EX 5 \& EX 8}$$

Therefore: $\left[\not{p} - m_0 - \Sigma(\not{p})\right]\Big|_{\not{p}=m} = 0$

$$\approx (\not{p} - m) \underbrace{\left(1 - \frac{d\Sigma}{d\not{p}}\Big|_{\not{p}=m}\right)}_{= 1/Z_2} + \mathcal{O}((\not{p}-m)^2)$$

now we can finally calculate

$$\delta m = m - m_0 = \Sigma_2(\not{p} = m) \approx \Sigma_2(\not{p} = m_0)$$

$$\delta m = \frac{\alpha}{2\pi} m_0 \int_0^1 dx \, (2-x) \log\left(\frac{x\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2}\right)$$

Diverges for $\Lambda \to \infty$:

$$\delta m \xrightarrow[\Lambda \to \infty]{} \frac{3\alpha}{4} m_0 \log\left(\frac{\Lambda^2}{m_0^2}\right)$$

We will address this issue when we discuss renormalisation.

$$\delta Z_2 = \frac{d\Sigma_2}{d\not{p}}\Big|_{\not{p}=m} = \frac{\alpha}{2\pi} \int_0^1 dx \left[-x \log \frac{x\Lambda^2}{(1-x)^2 m^2 + x\mu^2} + 2(2-x) \frac{x(1-x) m^2}{(1-x)^2 m^2 + x\mu^2}\right]$$

Also log UV divergence.

6.3. Dimensional Regularisation

Idea: making divergent integral convergent by introducing a parameter

$\lightning$ different ways to do this might give different results for observables $\rightsquigarrow$ regulator breaks symmetry

choose regulator which is compatible with postulated sym.

We already encountered Pauli-Villars reg.
gauge invariant but not covariant $\rightsquigarrow$ fails for QCD

$\hookrightarrow$ dimensional regularisation

Idea: compute integrals as an analytic function of dimension, i.e from last lecture

$$I = \int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2+\Delta)^2} = \underbrace{\int \frac{d\Omega_d}{(2\pi)^d}}_{A} \cdot \underbrace{\int_0^\infty d\, l_E \frac{l_E^{d-1}}{(l_E^2+\Delta)^2}}_{B}$$

A) $\int d\Omega_d = \dfrac{2\pi^{d/2}}{\Gamma(d/2)}$ [surface area of (d-1)-dim unit sphere]

B) $$= \frac{1}{2}\int_0^\infty d(l_E^2) \frac{(l_E^2)^{\frac{d}{2}-1}}{(l_E^2+\Delta)^2} = \frac{1}{2}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \int_0^1 dx\, x^{1-\frac{d}{2}}(1-x)^{\frac{d}{2}-1}$$

$x = \Delta/(l_E^2+\Delta)$

Trick: $\int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$$= \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$ expand around $d = 4+\varepsilon$

$$\Gamma\left(2-\frac{d}{2}\right) = \Gamma(\varepsilon/2) = \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon)$$

≈ 0.5772 cancles in Observables

$$\lim_{d\to 4} I = \frac{1}{(4\pi)^2}\left(\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi) + \mathcal{O}(\varepsilon)\right)$$

vs. Pauli-Villars from last lecture

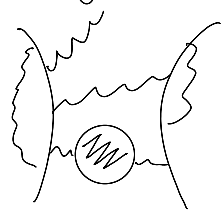
$$\lim_{\Lambda\to\infty} I = \frac{1}{(4\pi)^2}\left(\log x\Lambda^2 - \log\Delta + \mathcal{O}(\Lambda^{-1})\right)$$

7. Renormalisation

Task: give a physical interpretation for divergencies

7.1. Counting of UV divergences

Example QED:



$$\sim \int \frac{d^4k_1 \ldots d^4k_L}{(\not{k}_i - m)\ldots(k_j^2)}$$

Def.: superficial degree of divergence

$$D \equiv (\text{power of } k \text{ in numerator}) - (\text{power of } k \text{ in denominator})$$

$$= 4L - P_e - 2P_\gamma$$

P_γ: # photon propagators

L: # Loops

P_e: # electron propagators

Naive:
- Λ^D divergence for $D > 0$
- $\log\Lambda$ — " — $D = 0$
- no — " — $D < 0$

cutoff

⚡ often wrong

$$L = P_e + P_\gamma - V + 1$$

V: # vertices

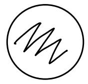
$$V = 2P_\gamma + N_\gamma = \frac{1}{2}(2P_e + N_e)$$

N_e: # external legs

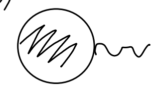
$$\boxed{D = 4 - N_\gamma - \frac{3}{2}N_e}$$

only depends on the number of external legs!

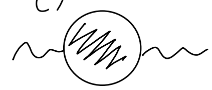
Candidates for "primitively" divergent integrals in QED:

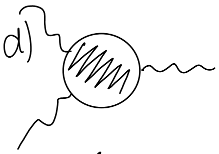
a)  $D=4$

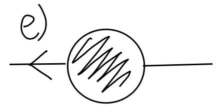
vacuum

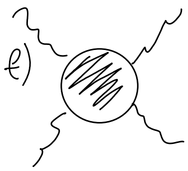
b)  $D=3$

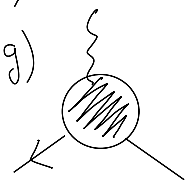
tad pole

c)  $D=2$

d)  $D=1$

e)  $D=1$

f)  $D=0$

g)  $D=0$

a) just shifts vacuum energy

b) $=0$ by Lorentz invariance

d) $=0$ —— " ——

f) is finite by symmetry

$\Rightarrow$ only 3 left, they renormalise the electron's e) mass and g) coupling to the em field
log div. log div.

and the polarisation of the vacuum c)