6. Gauge symmetries

Remember: Last lecture covariant ED.

$$S = 1/4 \int d^4x \, F_{\mu\nu} F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$$

Natural question: What are the symmetries of this action?

1) We already know Poincaré symmetry

$$x'^\mu = L^\mu{}_\nu x^\nu + a^\mu$$ But, how do fields transform?

- scalar: $$\boxed{\phi'(x') = \phi(x)}$$

infinitesimally: $x'^\mu = x^\mu + \xi^\mu(x)$ very small

$$\phi'(x+\xi) = \phi'(x) + \xi^\mu \partial_\mu \underbrace{\phi'(x)}_{= \phi(x) + \mathcal{O}(\xi)} + \mathcal{O}(\xi^2) = \phi(x)$$

$$\rightsquigarrow \quad \phi'(x) - \phi(x) := \boxed{\delta_\xi \phi(x) = -\xi^\mu \partial_\mu \phi(x)}$$

- one-form: $$A'_\mu(x') = A_\mu(x) - \partial_\mu \xi^\nu A_\nu(x)$$ rotation of tangent space

$$= A'_\mu(x) + \xi^\nu \partial_\nu A_\mu(x)$$

$$\rightsquigarrow \quad \boxed{\delta_\xi A_\mu(x) = -\xi^\nu \partial_\nu A_\mu(x) - \partial_\mu \xi^\nu A_\nu(x)}$$

in general $$\delta_\xi T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_r} = -\mathcal{L}_\xi T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_r}$$ Lie derivative

with $$\mathcal{L}_\xi T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = \xi^\rho \partial_\rho T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} - \partial_\rho \xi^{\mu_1} T^{\rho \mu_2 \dots \mu_r}{}_{\nu_1 \dots \nu_s} - \dots$$

$$- \partial_\rho \xi^{\mu_r} T^{\mu_1 \dots \mu_{r-1}\rho}{}_{\nu_1 \dots \nu_s} + \partial_{\nu_1} \xi^\rho T^{\mu_1 \dots \mu_r}{}_{\rho \nu_2 \dots \nu_s} + \partial_{\nu_s} \xi^\rho T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_{s-1}\rho}$$

derivative because of Leibniz rule

$$\mathcal{L}_\xi (U \cdot V) = \mathcal{L}_\xi U \cdot V + U \mathcal{L}_\xi V$$

Question: What happens to derivatives?

$$\delta_\xi (\partial_\mu A_\nu) = \partial_\mu \delta A_\nu$$

$$= -\mathcal{L}_\xi (\partial_\mu A_\nu) + \Delta_\xi (\partial_\mu A_\nu)$$ violation of covariant transformation

$$\Delta_\xi (\partial_\mu A_\nu) = \mathcal{L}_\xi (\partial_\mu A_\nu) - \partial_\mu (\mathcal{L}_\xi A_\nu)$$

$$= -\partial_\mu \partial_\nu \xi^\rho A_\rho$$

for Poincaré $\quad \xi^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \quad$ const.! $\quad$ (1)

$\rightsquigarrow \quad \partial_\mu \partial_\nu \xi^\rho = 0 \qquad$ even better for

$$\Delta_\xi (F_{\mu\nu}) = 2 \Delta_\xi (\partial_{[\mu} A_{\nu]}) = 2 \partial_{[\mu} \partial_{\nu]} \xi^\rho A_\rho = 0$$

finally, we need $\quad \mathcal{L}_\xi \eta_{\mu\nu} = \cancel{\xi^\rho \partial_\rho \eta_{\mu\nu}} + \partial_\mu \xi^\rho \eta_{\rho\nu} + \partial_\nu \xi^\rho \eta_{\mu\rho}$

$$\mathcal{L}_\xi \eta_{\mu\nu} = 2 \partial_{(\mu} \xi_{\nu)} \overset{(1)}{=} 2 \Lambda_{(\nu\mu)} = 0$$

because $\Lambda_{\mu\nu}$ is antisymm.

Summary: $\quad \mathcal{L}_\xi \eta_{\mu\nu} = 0, \quad \delta_\xi F_{\mu\nu} = -\mathcal{L}_\xi F_{\mu\nu}$

$$\delta_\xi (F_{\mu\nu} F^{\mu\nu}) = - \xi^\rho \partial_\rho (F_{\mu\nu} F^{\mu\nu})$$

therefore:
$$\delta_\xi S = +1/4 \int d^4x \, \xi^\rho \partial_\rho (F_{\mu\nu} F^{\mu\nu}) \checkmark$$
$$\overset{\text{i.b.p.}}{=} -1/4 \int d^4x \, \underbrace{\partial_\rho \xi^\rho}_{= \Lambda^\rho{}_\rho = 0} F_{\mu\nu} F^{\mu\nu} = 0$$

ED is invariant under Poincaré group.

2) Are there other symmetries? YES!

$$\boxed{A'_\mu(x) = A_\mu(x) + \partial_\mu \varepsilon(x)} \text{ arbitrary function}$$

Gauge transformation

$$\Rightarrow \delta_\varepsilon A_\mu(x) = A'_\mu(x) - A_\mu(x) = \partial_\mu \varepsilon(x)$$
$$\delta_\varepsilon F_{\mu\nu} = 2\partial_{[\mu} \delta_\varepsilon A_{\nu]} = 2 \partial_{[\mu} \partial_{\nu]} \varepsilon(x) = 0$$

again $$\delta_\varepsilon S = -1/4 \int d^4x \, 2 \underset{\overset{\parallel}{0}}{(\delta_\varepsilon F_{\mu\nu})} F^{\mu\nu} = 0$$

Summary:

Poincaré Symmetry

global

parameters $\Lambda^\mu{}_\nu$ and a^μ
$6 + 4 = 10$

do not depend on coordinates
$\hat{=}$ are everywhere the same

U(1)-Gauge symmetry

local

parameter $\varepsilon(x)$ depends on coordinates $\hat{=}$ can be different on each point

6.1. Gauge fixinging

Challenge: Some parts of $A_\mu(x)$ are redundant
$\rightsquigarrow$ What are the physical relevant ones?

$\rightsquigarrow$ Use gauge transformations to set parts of A_μ to zero.

$$A'_\mu = A_\mu + \partial_\mu \varepsilon \quad \underset{\sim}{\text{or}} \quad \partial^\mu A'_\mu = \partial^\mu A_\mu + \partial^\mu \partial_\mu \varepsilon$$

$$\rightarrow \boxed{\partial^\mu \partial_\mu \varepsilon = - \partial^\mu A_\mu}$$

a) <u>Lorentz gauge</u>

does not fully fix ε yet, alternative

$$A'^0 = A^0 + \overbrace{\partial^0 \varepsilon}^{\frac{\partial}{\partial t}} = 0$$

$$\rightarrow \boxed{\frac{\partial}{\partial t} \varepsilon = - A^0}$$

b) <u>Coulumb gauge</u>

both together imply: $\quad A_0 = A^0 = 0 \;\; \& \;\; \underbrace{\partial_i A^i = 0}_{\text{div } A = 0}$

$\rightsquigarrow$ only 2 of A_μ's 4 components are physical.

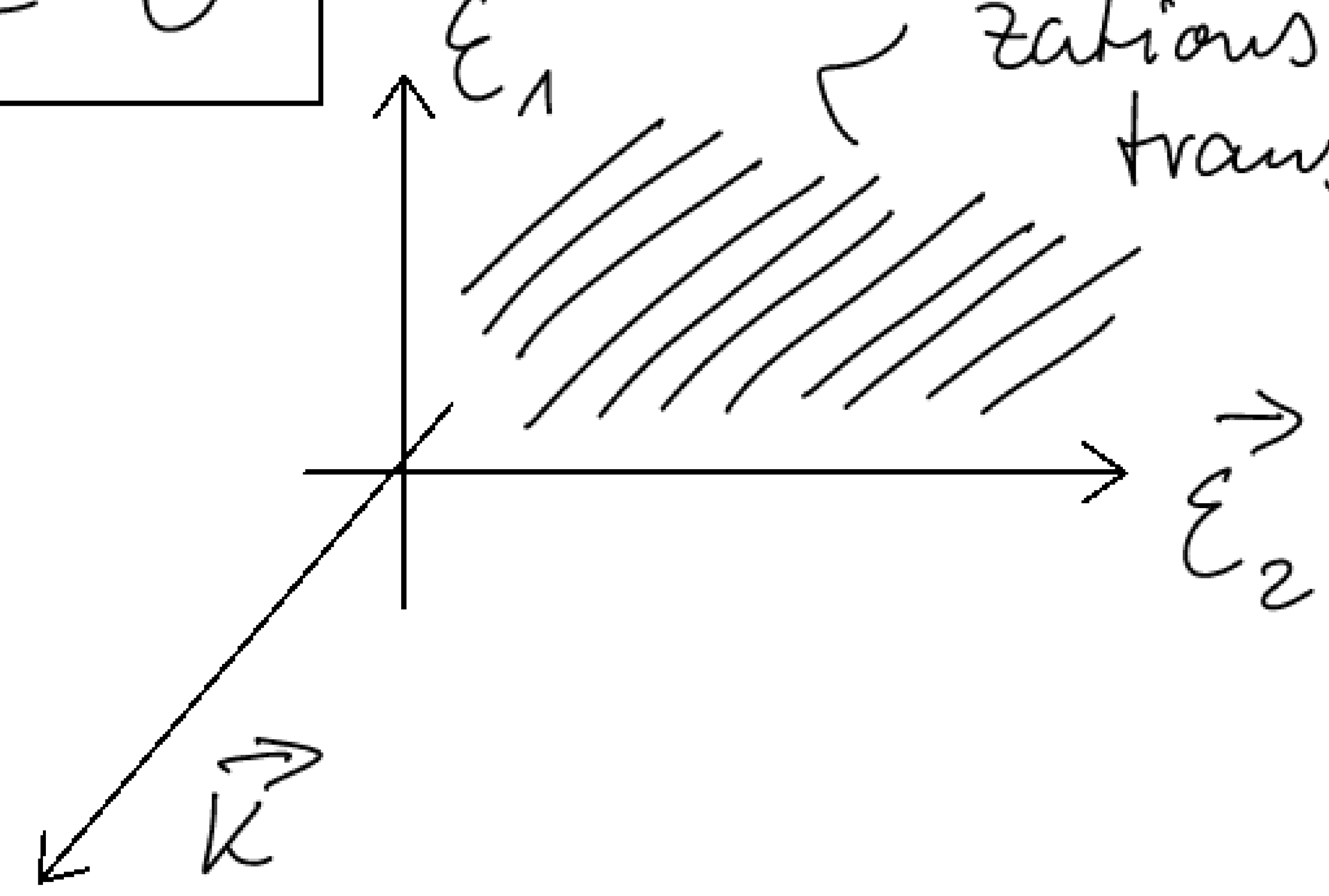
To see this look at the Fourier expansion

$$\vec{A}(\vec{x},t) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \sum_{\lambda=1}^{2} \vec{\varepsilon}_\lambda(\vec{k}) \left[a^\lambda_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} + (a^\lambda_k)^\dagger e^{-i\vec{k}\cdot\vec{x}} \right]$$

$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ as $A_0 = 0$ $\qquad$ polarization vectors

$$\partial_i A^i = 0 \iff \boxed{\vec{k} \cdot \vec{\varepsilon}^\lambda = 0}$$

$$\vec{\varepsilon}_\lambda \cdot \vec{\varepsilon}_s = \delta_{\lambda s}$$



allowed polarizations are transversal to wave vector