Remember from last lecture

$$\langle \mathcal{O}(x_1)\, \bar{\mathcal{O}}(x_2) \rangle = \frac{1}{(x_1-x_2)^{2\Delta}} \qquad \mathcal{O} = \text{scalar primary}$$

for general primaries

$$\langle \mathcal{O}^i(x)\, \bar{\mathcal{O}}_j(y) \rangle = \frac{C_{\mathcal{O}}}{(x-y)^{2\Delta}} \underbrace{D}_{\text{corresponding Lorentz representation}}(\overset{\text{Inversion}}{\tilde{I}}(x-y))^i{}_j$$

j: Lorentz index like $j=\mu$ or $j=\mu\nu$

example

$$\langle J_\mu(x)\, J_\nu(y) \rangle = \frac{C_J}{(x-y)^{2(d-1)}} I_{\mu\nu}(x-y)$$

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x^\mu x^\nu}{x^2} = D(I(x))_{\mu\nu}$$

Similar for 3-point function.

6.4. Conformal anomaly

classically $T^\mu{}_\mu = 0$ due to conformal sym.

after quantization, i.e. in $d=2$ $\boxed{\langle T^\mu{}_\mu \rangle = \frac{c}{24} R}$

remember the central charge

$$\frac{\delta}{\delta g^{\sigma\rho}} \rightsquigarrow \langle T^\mu{}_\mu(x)\, T_{\sigma\rho}(y) \rangle = -\frac{c}{12\pi}\left(\partial_\sigma \partial_\rho - \delta_{\sigma\rho}\partial^2\right)\delta^2(x-y)$$

$\neq 0$ even for flat space

There are no conformal anomalies in odd dimensions. But in 4d we have

$$\boxed{\langle T^\mu{}_\mu \rangle = \frac{c}{16\pi^2} C^{\mu\nu\sigma\rho} C_{\mu\nu\sigma\rho} - \frac{a}{16\pi^2} E}$$

Weyl tensor $\qquad$ Euler topological density

$$E = R^{\mu\nu\sigma\varsigma} R_{\mu\nu\sigma\varsigma} - 4 R^{\mu\nu} R_{\mu\nu} + R^2 \quad \text{and}$$

$$\int d^4 \sqrt{g}\, E = 4\pi \chi \sim \text{Euler number}$$

7. Supersymmetry

Consider fermionic symmetry generators.

7.1. Reminder on spinors in 4d

Using the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu$

γ- matrices $\qquad = -2\eta_{\mu\nu} \cdot \mathbb{1}$

we can write the Lorentz generators as $J^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

in particular $(\gamma_0)^2 = \mathbb{1}$ and $(\gamma_i)^2 = -\mathbb{1}$

$\rightsquigarrow$ we can choose $\gamma_0^\dagger = \gamma_0$ and $\gamma_i^\dagger = -\gamma_i$

One can check that $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ satisfies

$$\{\gamma_5, \gamma_\mu\} = 0, \quad \underbrace{\gamma_5^2 = \mathbb{1}, \quad \gamma_5 = \gamma_5^\dagger}_{\text{eigenvalues are } \pm 1}$$

$\Downarrow$

diagonalize

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\beta} \\ \bar\sigma^{\mu\dot\alpha\beta} & 0 \end{pmatrix} \qquad \begin{matrix} \alpha = \{1,2\} & \sigma^\mu = (-\mathbb{1}_{2\times 2}, \sigma^i) \\ \dot\alpha = \{1,2\} & \bar\sigma^\mu = (-\mathbb{1}_{2\times 2}, -\sigma^i) \end{matrix}$$

$\sim$ fund. of $SL(2,\mathbb{R})$

$$SO(3,1) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$$

We now have spinors like

$$\Psi = \begin{pmatrix} \psi_\beta \\ \bar\psi^{\dot\beta} \end{pmatrix}, \text{ with } \gamma^\mu \Psi = \begin{pmatrix} \sigma^\mu_{\alpha\dot\beta}\bar\psi^{\dot\beta} \\ \bar\sigma^{\mu\dot\alpha\beta}\psi_\beta \end{pmatrix} \quad \text{and}$$

charge conjugation $C = \begin{pmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & \varepsilon_{\dot\alpha\dot\beta} \end{pmatrix}$ such that $(C\psi)^{\dagger}\psi = \bar\psi\psi$

$\varepsilon_{12} = 1$

7.2. SUSY algebra

The new generators we want to add to Poincaré are

$$Q^a = \begin{pmatrix} Q^a_\alpha \\ \bar{Q}^{a\dot\alpha} \end{pmatrix} \qquad a = 1, \ldots, \mathcal{N}$$

$\mathcal{N}$ number of independent supersymmetries

$\mathcal{N}=1$ algebra

⚡ We are working with a super Lie algebra.

$$[T_1, T_2\} = T_1 \cdot T_2 - (-1)^{g_1 g_2} T_2 T_1$$

$g_1 g_2$: grading of generators $\begin{cases} 0 & \text{for } P, J \\ 1 & \text{for } Q \end{cases}$

$[Q_\alpha, J^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta$, $\quad [\bar{Q}_{\dot\alpha}, J^{\mu\nu}] = \varepsilon_{\dot\alpha\dot\beta}(\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\gamma}\bar{Q}^{\dot\gamma}$,

$[Q_\alpha, P^\mu] = 0$, $\quad [\bar{Q}_{\dot\alpha}, P^\mu] = 0$,

and $\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu$, while

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \qquad + \text{Poincaré}$$

this algebra has an additional $U(1)$ symmetry

$$Q_\alpha \to Q'_\alpha = e^{i\lambda} Q_\alpha \quad \text{and} \quad \bar{Q}_{\dot\alpha} \to \bar{Q}'_{\dot\alpha} = e^{-i\lambda}\bar{Q}_{\dot\alpha}$$

called R-symmetry it is generated by R with

$$[Q_\alpha, R] = Q_\alpha \quad \text{and} \quad [\bar{Q}_{\dot\alpha}, R] = -\bar{Q}_{\dot\alpha}$$

$\mathcal{N}>1$

$$\{Q^a_\alpha, \bar{Q}_{b\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^a_b \quad \text{and}$$

$$\{Q^a_\alpha, Q^b_\beta\} = \varepsilon_{\alpha\beta} Z^{ab}, \quad \{\bar{Q}_{a\dot\alpha}, \bar{Q}_{b\dot\beta}\} = \varepsilon_{\dot\alpha\dot\beta} \bar{Z}_{ab}$$

central charges, generate the center of the algebra

$Z^{ab} = Z^{[ab]}$ and $Z^{ab} = (\bar{Z}^{+})_{ab}$

again we have R-symmetry, by

$$Q^a_\alpha \to Q'^a_\alpha = R^a{}_b Q^b_\alpha , \qquad \bar{Q}_{a\dot{\alpha}} \to \bar{Q}'_{a\dot{\alpha}} = \bar{Q}_{b\dot{\alpha}} (R^{+})^b{}_a$$

such that $R^a{}_c R^b{}_d Z^{cd} = Z^{ab}$

for $Z^{ab} = 0$ $U(N)$ otherwise subgroup

7.3. Representations

Like $\vec{L}^2$ for $SU(2)$

can be sorted by eigenvalues of Casimir Operator.

There are two $P^2 = P_\mu P^\mu$ (rest mass) and

$$W^2 = \tilde{C}_{\mu\nu} \tilde{C}^{\mu\nu}$$ more in the tutorial

General properties: In a representation

- mass of all fields is the same
- gauge sym. commutes with SUSY algebra
  - $\leadsto$ all fields have the same gauge rep.
- the number of bosons and fermions is the same

massless representations

from Poincaré-algebra, we know $P_\mu P^\mu = 0$

$\leadsto P_\mu = (E, \underbrace{0, 0,}_{\text{invariant under } SO(2) \text{ rotations}} E)$ in rest frame

" little group

$SO(2)$ has one generator, J_{12}. Its eigenvalues λ are called helicity.

$\leadsto |P_\mu, \lambda\rangle$

acting on these states we get

$$\{Q^a_\alpha, \bar{Q}_{b\dot{\beta}}\} |p_\mu, \lambda\rangle = 4\delta^a_b \, E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} |p_\mu, \lambda\rangle$$

and define the fermionic ladder operators

$$a^b = \frac{Q^b_1}{2\sqrt{E}} \quad \text{and} \quad a^+_b = \frac{\bar{Q}_{b\dot{1}}}{2\sqrt{E}} \quad \text{satisfying}$$

$$\{a^b, a^+_c\} = \delta^b_c, \quad \{a^b, a^c\} = \{a^+_b, a^+_c\} = 0$$

form $[Q^a_\alpha, J_{12}]$ we also see that

$$a^b |p_\mu, \lambda\rangle = |p_\mu, \lambda - 1/2\rangle$$

↗ a^b = lowering op

a^+_b = raising -"-

Each multiplet starts from lowest helicity state

$|\Omega\rangle$ and contains $\{|\Omega\rangle, a^+_a|\Omega\rangle, a^+_b a^+_a|\Omega\rangle, \ldots\}$

" $|p_\mu, \lambda_0\rangle$ $\quad a \neq b$

in total 2^N fields

$\lambda_0 = 0$ $\quad N=1$ chiral multiplet

$\lambda_0 = 1/2$ $\quad$ vector -"- $\quad$ ....