8. Representations

In physics we don't see Lie algebra directly, but only its consequences ⇝ representation

i.e. $[t_a, t_b] = f_{ab}{}^c t_c$

task: map t_a, t_b and $[\cdot,\cdot]$ to known structures like matrices A, B and $[A,B] = A \cdot B - B \cdot A$

matrix multipl.

8.1. Module

Def.: Take a Lie algebra g and a linear vector space V over a field $\mathbb{F}$. A representation ρ of g is a linear mapping:

$\rho(x): V \to V$ for $\forall\, x \in g$

such that: $\boxed{\rho(x) \circ \rho(y) - \rho(y) \circ \rho(x) = \rho([x,y])}$

remarks: - $\circ$ is the usual composition i.e.

$\rho(x) \circ \rho(y)[v] = \rho(x)[\rho(y)[v]]$, $v \in V$

- V is a vector space & ρ is linear ⇝ $\rho: g \to gl(n, \mathbb{F})$

→ $\rho(x)$ can be thought of as $n \times n$ matrix over the field $\mathbb{F}$ — dimension of V

$\rho: x \to gl(V)$ is an homomorphism of g

structure preserving map

V is called the representation space or module (V, g)

A module is also equipped with another operation

$\bullet: g \times V \to V$

$(x, v) \mapsto x \bullet v = w \in V$

not the composition!

such that for $a, b \in \mathbb{F}$, $x, y \in g$ and $v, w \in V$

1) $(ax + by) \bullet v = a(x \bullet v) + b(y \bullet v)$,
2) $x \bullet (av + bw) = a(x \bullet v) + b(x \bullet w)$,
3) $[x, y] \bullet v = x \bullet (y \bullet v) - y \bullet (x \bullet v)$ hold

$$\boxed{\underbrace{x \bullet v}_{\text{module}} = \underbrace{\varrho(x)[v]}_{\text{representation}}}$$

⇝ in physics literature a module is often called a representation.

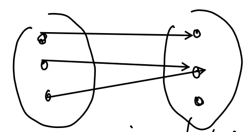
Properties of modules:

I) two modules (V, g) and (W, g) for the same Lie algebra g and a linear mapping $f: V \to W$ such that

$$\varrho_W(x) \circ f = f \circ \varrho_V(x)$$

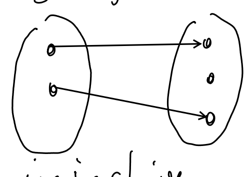
f intertwines the action of g and is called an intertwiner

II) If f is surjective, W is called the homomorphic image of V.



surjective and

III) If f is also injective, then (V, g) and (W, g) are isomorphic modules. They are indistinguishable.



injective

one-to-one

IV) A representation is called faithful or effective if $x, y \in g$ with $x \neq y \Rightarrow \varrho(x) \neq \varrho(y)$.

8.2 Constructing representations

Starting from a single representation ϱ of g acting on the module V, one can generate other representations by the following operations from linear algebra:

I) conjugation: $\bar{\varrho}(x) := -\varrho^T(x)$

works because $[t_a, t_b]^T = [t_b^T, t_a^T] = -[t_a^T, t_b^T]$

$$-(f_{ab}{}^{c}\, t_c)^T = f_{ab}{}^{c}\, t_c^T$$

$\Leftrightarrow \quad [-t_a^T, -t_b^T] = f_{ab}{}^{c}(-t_c^T)$ //

This representation acts on the dual vector space V^*.

II) direct sum: Take two g modules, V and W, then we can construct the direct sum module $V \oplus W$ with:

$$(\rho_V \oplus \rho_W)(x)[v \oplus w] := \rho_V(x)[v] \oplus \rho_W(x)[w]$$

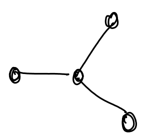
III) tensor product:

$$(\rho_V \otimes \rho_W)(x)[v \otimes w] := \rho_V(x)[v] \otimes w + v \otimes \rho_W(x)[w]$$

IV) endomorphism: For an homomorphism of the Lie algebra $\varphi: g \to g$ we can construct

preserves brackets

$$\rho^{\varphi}(x) = \rho(\varphi(x))$$

Example: $so(8)$  has three fold symmetry

$\hookrightarrow$ three 8 dimensional representations

8_s, 8_c, $8_v \leftarrow$ vector called triality of $so(8)$

spinor, conjugate spinor

8.3 Irreducible representations

These constructions lead to many representations. But they might decompose into smaller representations.

Remember: 1^{st} lecture product representation of 3-dimensional representation of $su(2)$

$$3 \otimes 3 \to 1 \oplus 3 \oplus 5$$

Question: What are the smallest representations one can get from a given representation?

Answer: Irreducible representations = irreps

Def.: A submodule U of a module V is a subvector-space $U \subset V$ such that:

$$\rho(x)[U] \subseteq U \quad \text{for } \forall\, x \in g.$$

Each module has two trivial submodules:

a) the module itself and

b) the zero vector space $\{0\}$.

$\boxed{\text{An irrep does not possess any other submodules than these two trivial ones.}}$

⇝ A reducible representation contains non-trivial submodules.

Schur's lemma: Any intertwiner f between two irreducible modules V and W is either an isomorphismus or zero ($f = 0$).

A module is called fully reducible if it can be written as a direct sum of irreps.

Any Lie algebra has always the following three irreps:

a) trivial representation: $\rho(X) = 0$

But we can't learn much about g from it.

b) adjoint representation: $ad_X(Y) = [X, Y],\ X, Y \in g$

Here $V = g$ and we have used it extensively before.

c) fundamental representation:

From it all other reps can be obtained with the tools from section 8.2.