4.9. Highest weight representations

We know irreps for $su(N)$ from Young tableaux.

Question: What about other simple Lie algebras?

4.9.1. Highest weight irreps of $su(2)$

three generators: H, E_+, E_-

Cartan sub alg. positive/negative roots

$$[H, E_\pm] = \pm 2 E_\pm \,, \qquad [E_+, E_-] = H \quad (\text{see } 4.5)$$

Irreps are characterized by highest weight vectors

V_Λ with
$$\boxed{\begin{aligned} H V_\Lambda &= \Lambda V_\Lambda \\ E_+ V_\Lambda &= 0 \end{aligned}}$$

Check: $H(E_- V_\Lambda) = E_- H V_\Lambda - 2 E_- V_\Lambda = (\Lambda - 2) E_- V_\Lambda$

$$\begin{array}{c} \boxed{V_\Lambda} \\ \downarrow E_- \\ \boxed{V_{\Lambda-2}} \\ \downarrow E_- \\ \vdots \end{array}$$

$$V_{\Lambda-2} = E_-^n$$

$$\begin{aligned} E_+ V_{\Lambda-2} &= E_+ E_- V_{\Lambda-2n+2} \\ &= \big([E_+, E_-] + E_- E_+ \big) V_{\Lambda-2n+2} \\ &= \big(H + \underbrace{E_- E_+}_{?} \big) V_{\Lambda-2n+2} \end{aligned}$$

We know $? \sim V_{\Lambda-2n+2}$

$$\begin{aligned} E_- E_+ V_{\Lambda-2n} &= r_n V_{\Lambda-2n} \\ E_- E_+ V_{\Lambda-2n+2} &= r_{n-1} V_{\Lambda-2n+2} \\ H V_{\Lambda-2n+2} &= (\Lambda - 2n + 2) V_{\Lambda-2n+2} \end{aligned}$$

E_- on both sides

$$\rightsquigarrow \quad r_n \cancel{V_{\Lambda-2n}} = (\Lambda - 2n + 2 + r_{n-1}) \cancel{V_{\Lambda-2n}}$$

$$r_n = \Lambda - 2n + 2 + r_{n-1} \quad \text{with } r_0 = 0 \quad (E_+ V_\Lambda = 0)$$

Solve with ansatz $\quad r_n = \alpha n^2 + \beta n$

$\alpha = -1,\ \beta = \Lambda + 1: \quad r_n = n(\Lambda + 1 - n)$ and

$$r_{\Lambda+1} = 0$$

$$\leadsto \boxed{E_- \underbrace{V_{\Lambda - 2\Lambda}}_{V_{-\Lambda}} = 0} \quad \underline{\text{lowest weight vector}}$$

$$\begin{tikzcd} \boxed{V_\Lambda} \arrow[d, shift right, "E_-"'] \\ \boxed{V_{\Lambda-2}} \arrow[u, shift right, "E_+"'] \arrow[d, shift right, "E_-"'] \\ \vdots \arrow[u, shift right, "E_+"'] \arrow[d, shift right, "E_-"'] \\ \boxed{V_{-\Lambda}} \arrow[u, shift right, "E_+"'] \end{tikzcd}$$

<u>unitary representation:</u> $\quad E_+^\dagger = E_-$

$\nearrow$ angular momentum in QM:

$H \sim L_z,\quad E_+ \sim L_+,\quad E_- \sim L_-$ and we also have

$$\vec{L}^2 = 1/4\, H(H+2) + E_- E_+, \quad \vec{L}^2 V_\Lambda = 1/4\, \Lambda(\Lambda+2) V_\Lambda$$

$\leadsto$ spin $j = 1/2\, \Lambda$

4.9.2 Highest weight representation

For $su(2)$, we found $\quad V(\Lambda) = \bigoplus_{\lambda = -\Lambda} \underbrace{\{ X_\lambda V_\lambda \mid X_\lambda \in \mathbb{C} \}}_{\bar{V}_\lambda}$ labeled by Cartan generator's eigen values

generalization: $\quad V = \bigoplus_{\{\lambda\}} \bar{V}_\lambda \quad$ with $\quad H_i V_\lambda = \lambda^i V_\lambda$

$\rightarrow$ <u>weight</u> vector $\lambda = (\lambda^1 \ldots, \lambda^r)$, $r = \text{rank}\, g$

The collection of all weights is called <u>weight system</u>. It extends the root system, which only contains the weights of the adjoint representation.

⚠ Not all $\bar{V}_\lambda$'s for generic representations are one-dimensional (like for $su(2)$).

Same normalization as for roots in 4.5.

$$\lambda^i = \lambda(H_i) = (\alpha_i^\vee, \lambda) \in \mathbb{Z}$$

simple co-root $\nearrow$ with dual <u>fundamental weights</u> Λ_i:

$$(\Lambda_i, \alpha_j^\vee) = \delta_{ij}$$

$\rightsquigarrow$ any weight can be written as $\lambda = \lambda^i \Lambda_i$ ← <u>Dynkin labels</u>

Generalization of su(2) irreps

$$\boxed{\begin{array}{l}\text{For any finite dimensional, irrep of } g \text{ there is a} \\ \text{highest weight such that} \\ \qquad E_\alpha V_\Lambda = 0 \qquad \forall \alpha > 0 \text{ (positive roots)} \\ \text{where } V_\Lambda \text{ is one dimensional}\end{array}}$$

- all other weights are obtained by acting with negative root (lowering operator)

  $$H_i E_{-\alpha} V_\lambda = \dots = (\lambda - \alpha)^i E_\alpha V_\lambda$$

  $\Rightarrow$ any root can be written as $\lambda = \Lambda - \beta$,

  $$\beta = \sum_i n^i \alpha_i \quad , n_i \in \mathbb{N}$$

- $\sum_i n^i$ = <u>level</u> or <u>depth</u> of the weight

- If a weight appears n-times in this process, it has <u>multiplicity</u> n.

  $$\mathrm{mult}_\Lambda(\lambda) = n \quad \rightsquigarrow \quad \dim_{\mathbb{C}}(V_\lambda) = n$$

Freudental reduction formula

$$\mathrm{mult}_\Lambda(\lambda) = \frac{2 \sum\limits_{\alpha>0} \sum\limits_{\substack{m>0 \\ \text{and } \lambda + m\alpha \in V_\Lambda \text{ weight system}}} (\lambda + m\alpha, \alpha)\, \mathrm{mult}_\Lambda(\lambda + m\alpha)}{(\Lambda + \rho, \Lambda + \rho) - (\lambda + \rho, \lambda + \rho)}$$

$$\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$$

- For $\lambda^i > 0$, we can substract λ^i-times the root α^i. We stop, when no further α_i's can be substracted.

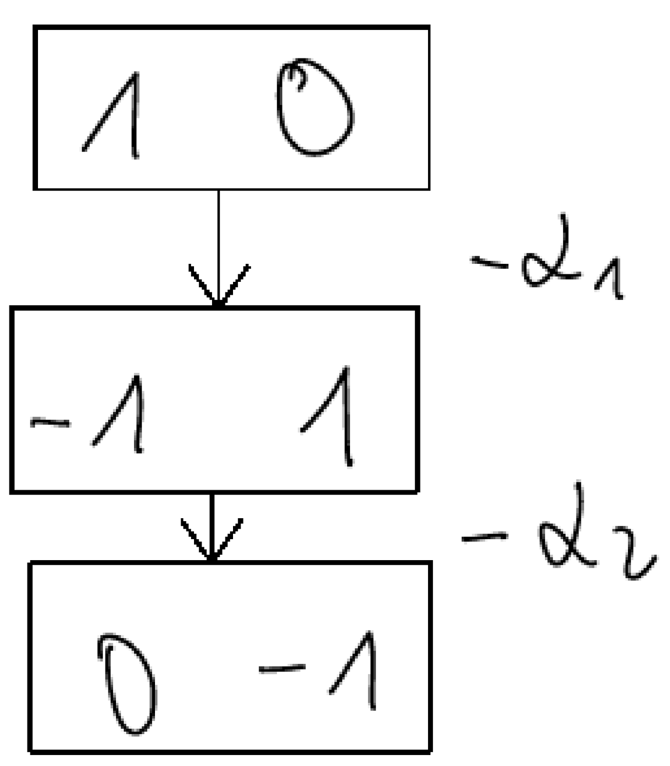
- The highest weight of the conjugate irrep $\bar{V}$ is $\Lambda_{\bar{V}} = -\lambda_{min}$ ← the lowest weight of V.

→ if $\Lambda = -\lambda_{min}$ V is called <u>self conjugated</u>

Example su(3)

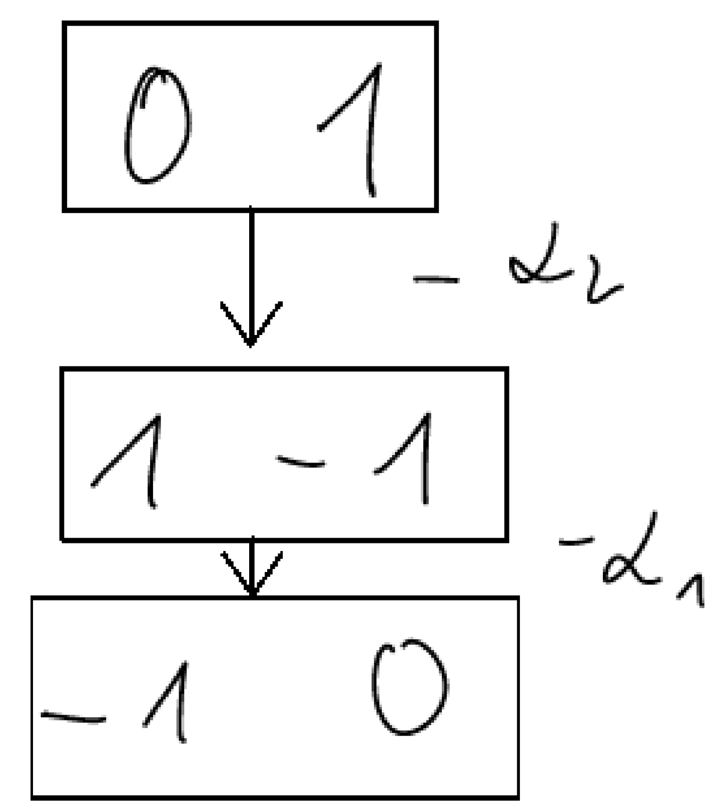
remember simple roots: $\alpha_1 = \boxed{2 \quad -1}$, $\alpha_2 = \boxed{-1 \quad 2}$

$\Lambda_{\square} =$



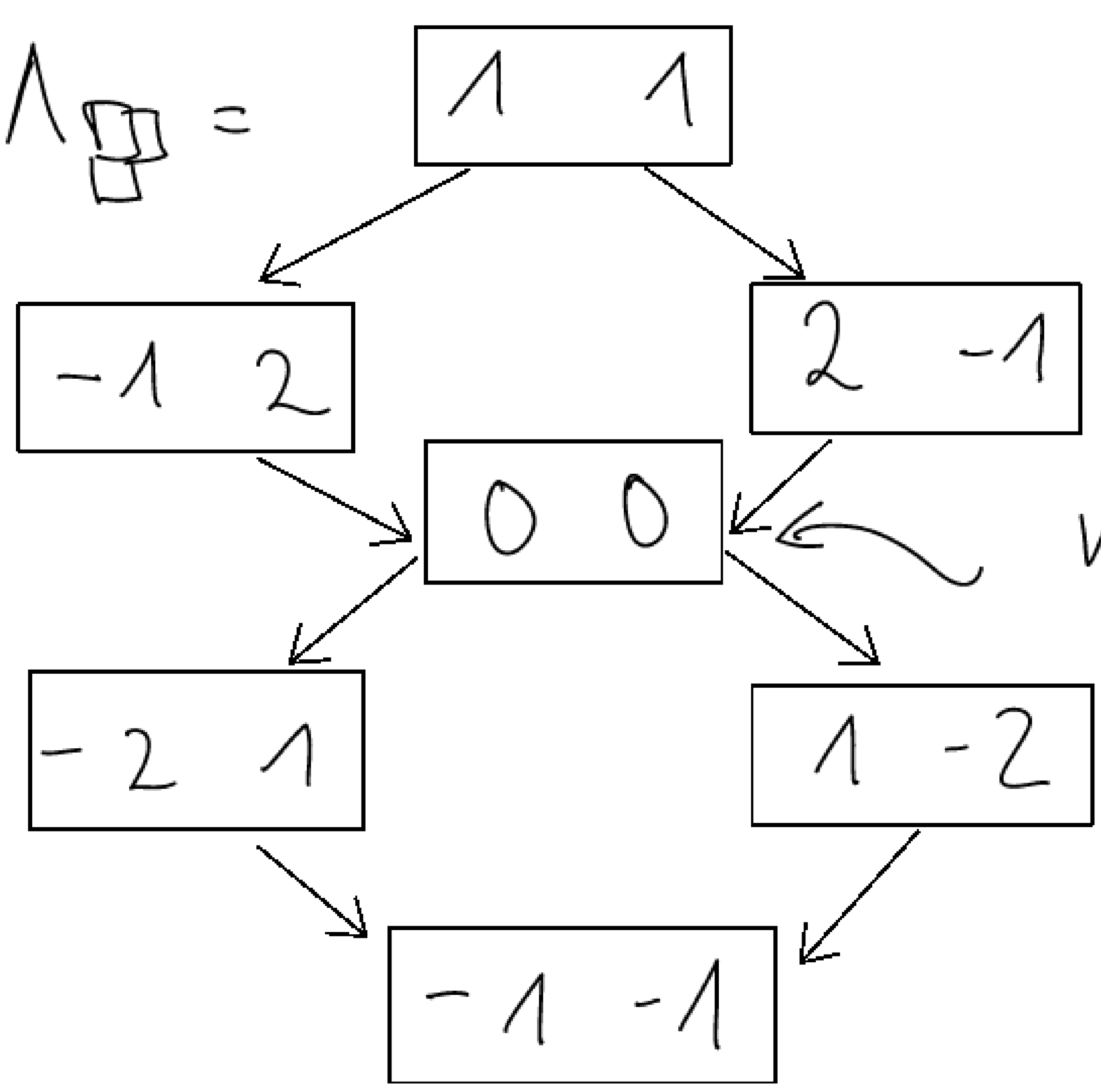
$\boxed{3}$

$\Lambda_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} =$



$\begin{smallmatrix}\boxed{3}\\ \boxed{2}\end{smallmatrix}$ $\quad \frac{3\cdot 2}{2} = 3$ ✓

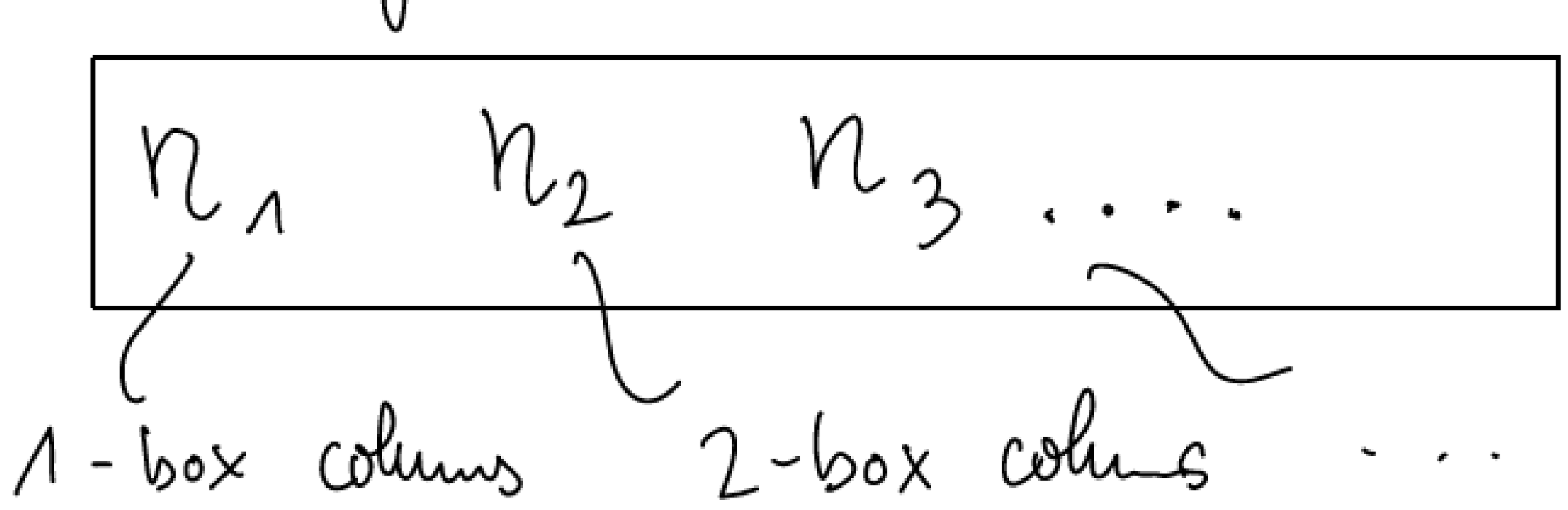
$\Lambda_{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}} =$



$\begin{smallmatrix}\boxed{3}\boxed{4}\\ \boxed{2}\end{smallmatrix}$ $\quad \frac{2\cdot \cancel{3}\cdot 4}{\cancel{3}} = 8$

$\mathrm{mult}_{\boxed{1\,1}}(\boxed{0\ 0}) = 2$

one-to-one correspondence between highest weight and Young diagram.



i.e. $\boxed{2 \quad 1 \quad 2}$ ⟹