6. Loop Effects

Motivation: We have the path integral & Feynman diagrams. Time to do some physics!

Remember EX 5: physical mass m_{phys} ($\neq m$ in Lagrangian) from 1-particle irreducible (1PI) function. Today more details.

6.1. Field-Strength Renormalisation

Take Hamiltonian of interacting scalar theory which is invariant under space and time translations.

Conserved charges: $P_\mu \xrightarrow[\text{quantisation}]{}$ operator $\hat{P}_\mu$

$[\hat{P}_\mu, \hat{H}] = 0 \rightarrow$ have same eigenstates $p_\mu |\lambda_p\rangle = \hat{P}_\mu |\lambda_p\rangle$

Completeness: $\mathbb{1} = |0\rangle\langle 0| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} |\lambda_p\rangle\langle\lambda_p|$

$E_p(\lambda) = \sqrt{|\vec{p}|^2 + m_\lambda^2}$ energy of state $|\lambda_p\rangle$ with mass m_λ

For $x^0 > y^0$:

$$\langle 0|\phi(x) \overset{\mathbb{1}}{\swarrow} \phi(y)|0\rangle = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} \langle 0|\phi(x)|\lambda_p\rangle\langle\lambda_p|\phi(y)|0\rangle$$

$$\langle 0|\phi(x)|\lambda_p\rangle = \langle 0| e^{i\hat{P}x} \phi(0) e^{-i\hat{P}x} |\lambda_p\rangle$$

$$= \langle 0|\phi(0)|\lambda_p\rangle e^{-ipx} \Big|_{p^0 = E_p}$$

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$\phi(0) = U_p \phi(0) U_p^{-1}$ & $\quad |\lambda_p\rangle = U_p |\lambda_0\rangle$

$\langle 0| U_p = \langle 0|$ $\quad$ Lorentz boost

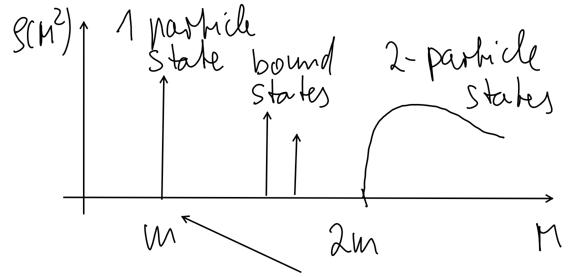
(vacuum is Lorentz invariant)

$$= \sum_\lambda \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_\lambda^2 + i\epsilon} e^{-ip(x-y)} |\langle 0|\phi(0)|\lambda_0\rangle|^2$$

Same for $y^0 > x^0 \rightsquigarrow$ $\boxed{\langle 0|T\phi(x)\phi(y)|0\rangle = \int \frac{dM^2}{2\pi} \rho(M^2) D_F(x-y, M^2)}$

Källén - Lehmann spectral representation with positive spectral density function

$$\rho(M^2) = \sum_\lambda (2\pi)\,\delta(M^2 - m_\lambda^2)\,|\langle 0|\phi(0)|\lambda_0\rangle|^2$$



All this just from symmetry !

Similar for higher spins.

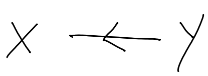
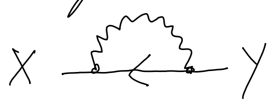
$$\rho(M^2) = 2\pi\,\delta(M^2 - m^2)\cdot \underset{\nwarrow}{Z} + \text{rest}$$

field strength renormalisation

Compare to EX 5: $m = m_{phys}$ and the mass parameter in Lagrangian is m_0 = bare mass

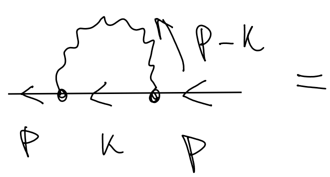
6.2. Electron Self-Energy

So far only symmetries, now Feynman diagrams in QED

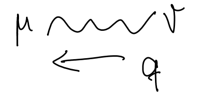
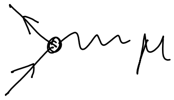
$\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle =$  +  + ...

with the free field propagator  $= \dfrac{i\not p + m_0}{p^2 - m_0^2 + i\varepsilon}$

the second diagram evaluates to

 $= \dfrac{i(\not p + m_0)}{p^2 - m_0^2}\left[-i\Sigma_2(p)\right]\dfrac{i(\not p + m_0)}{p^2 - m_0^2}$ with

$$-i\Sigma_2(p) = (-ie)^2\int\frac{d^4k}{(2\pi)^4}\,\gamma^\mu\,\frac{i(\not k + m_0)}{k^2 - m_0^2 + i\varepsilon}\,\gamma_\mu\,\frac{-i}{(p-k)^2 - \boxed{\mu^2} + i\varepsilon}$$

(Remember  $= \dfrac{-ig_{\mu\nu}}{q^2 + i\varepsilon}$ and  $= -ie\gamma^\mu$)

μ = small mass for the photon, to remove the divergence at $(p-k)^2 = 0$ (IR divergence).

Trick: Feynman parameter $\dfrac{1}{A\cdot B} = \displaystyle\int_0^1 \frac{dx}{(xA + (1-x)B)^2}$

$$-i\Sigma_2(p) = -e^2\int_0^1 dx\int\frac{d^4l}{(2\pi)^4}\,\frac{-2x\not p + 4m_0}{(l^2 - \Delta + i\varepsilon)^2}$$

$\ell = k - xp$ and $\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$

This integral is still divergent! We have to regularise it (details next lecture). Here Pauli-Villars reg.:

$$\boxed{\int \frac{d^4\ell}{2\pi}\frac{1}{(\ell^2-\Delta)^2} \to \frac{i}{(4\pi)^2}\int_0^\infty d\ell_E^2 \left(\frac{\ell_E^2}{(\ell_E^2+\Delta)^2} - \frac{\ell_E^2}{(\ell_E^2+\Delta_\Lambda)^2}\right)}$$

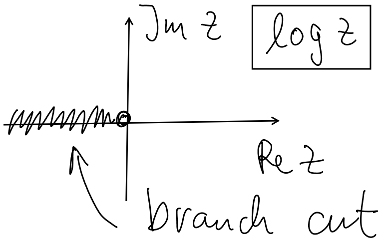
$$= \frac{i}{4\pi}\log\left(\frac{\Delta_\Lambda}{\Delta}\right)$$

- Λ = mass of photon, like μ, but we send $\Lambda \to \infty$

  $\to \Delta_\Lambda = -x(1-x)p^2 + x\Lambda^2 + (1-x)m_0^2 \xrightarrow[\Lambda\to\infty]{} x\Lambda^2$
- ℓ_E = Wick rotated 4-momentum $\ell_E^0 = -i\ell^0$

results in $\Sigma_2(p) = \frac{\alpha}{2\pi}\int_0^1 dx\,(2m_0 - x\not{p})\log\left(\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2}\right)$

$\alpha = \frac{e^2}{4\pi}$ = fine structure constant



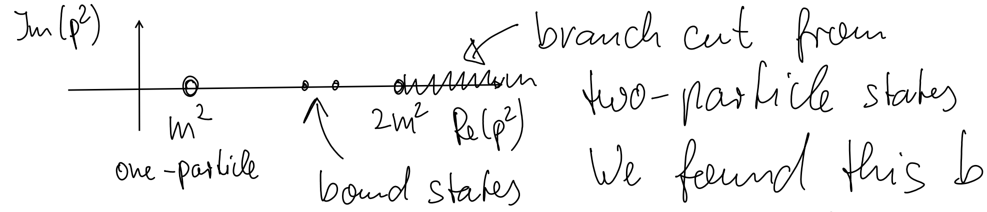
begins at $(1-x)m_0^2 + x\mu^2 - x(1-x)p^2 = 0$

$$x_{1/2} = \frac{1}{2} + \frac{m_0^2-\mu^2}{2p^2} \pm \frac{1}{2p^2}\sqrt{[p^2-(m_0+\mu)^2][p^2-(m_0-\mu)^2]}$$

remember $0 \le x \le 1$ ⇝ only on solution $p^2 = (m_0+\mu)^2$

This is the threshold to create two particles (electron + photon).

Compare with FT of $\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle$:



We found this branch cut

Question: What about pole at m^2?

Answer: We need to sum an infinite series of diagrams.